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HOMOGENEOUS SPACES DEFINED BY LIE GROUP AUTOMORPHISMS. II

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7. Noncompact coset spaces defined by automorphisms of order 3

We will drop the compactness hypothesis on G in the results of §6, doing this in such a way that problems can be reduced to the compact case. This involves the notions of reductive Lie groups and algebras and Cartan involutions.

Let $\mathfrak{G}$ be a Lie algebra. A subalgebra $\mathfrak{K} \subset \mathfrak{G}$ is called a *reductive subalgebra* if the representation $ad_{\mathfrak{G}}|_{\mathfrak{K}}$ of $\mathfrak{K}$ on $\mathfrak{G}$ is fully reducible. $\mathfrak{G}$ is called *reductive* if it is a reductive subalgebra of itself, i.e. if its adjoint representation is fully reducible. It is standard ([11, Theorem 12.1.2, p. 371]) that the following conditions are equivalent:
(7.1a) $\mathfrak{G}$ is reductive,
(7.1b) $\mathfrak{G}$ has a faithful fully reducible linear representation, and
(7.1c) $\mathfrak{G} = \mathfrak{G}' \oplus \mathfrak{Z}$, where the derived algebra $\mathfrak{G}' = [\mathfrak{G}, \mathfrak{G}]$ is a semisimple ideal (called the "semisimple part") and the center $\mathfrak{Z}$ of $\mathfrak{G}$ is an abelian ideal.

Let $\mathfrak{G} = \mathfrak{G}' \oplus \mathfrak{Z}$ be a reductive Lie algebra. An automorphism σ of $\mathfrak{G}$ is called a *Cartan involution* if it has the properties (i) $\sigma^2 = 1$ and (ii) the fixed point set $\mathfrak{G}'^{\sigma}$ of $\sigma|_{\mathfrak{G}'}$ is a maximal compactly embedded subalgebra of $\mathfrak{G}'$. The whole point is the fact ([11, Theorem 12.1.4, p. 372]) that

(7.2) *Let $\mathfrak{K}$ be a subalgebra of a reductive Lie algebra $\mathfrak{G}$. Then $\mathfrak{K}$ is reductive in $\mathfrak{G}$ if and only if there is a Cartan involution σ of $\mathfrak{G}$ such that $\sigma(\mathfrak{K}) = \mathfrak{K}$.*

Let G be a Lie group. We say that G is *reductive* if its Lie algebra $\mathfrak{G}$ is reductive. Let K be a Lie subgroup of G. We say that K is a *reductive subgroup* if its Lie algebra $\mathfrak{K}$ is a reductive subalgebra of $\mathfrak{G}$. Let σ be an automorphism of G. We say that σ is a *Cartan involution* of G if σ induces a Cartan involution of $\mathfrak{G}$.

Let G be a reductive Lie group, and K a closed reductive subgroup such that G acts effectively on $X = G/K$. Choose a Cartan involution σ of $\mathfrak{G}$ which preserves $\mathfrak{K}$, and consider the decomposition into (± 1)-einspaces of σ:

$$(7.3a) \qquad \mathfrak{G} = \mathfrak{G}^{\sigma} + \mathfrak{M}, \qquad \mathfrak{K} = \mathfrak{K}^{\sigma} + (\mathfrak{K} \cap \mathfrak{M}).$$

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That decomposition defines compact real forms of $\mathfrak{G}^C$ and $\mathfrak{K}^C$:

$$(7.3b) \qquad \mathfrak{G}_u = \mathfrak{G}^\sigma + \sqrt{-1}\,\mathfrak{M}\,, \qquad \mathfrak{K}_u = \mathfrak{K}^\sigma + \sqrt{-1}\,(\mathfrak{K} \cap \mathfrak{M})\,.$$

7.4. Lemma. *There is a unique choice of compact connected Lie group G_u with Lie algebra $\mathfrak{G}_u$ which has the properties [Z_u denotes the identity component of the center of G_u]*

(i) *the analytic subgroup K_u for $\mathfrak{K}_u$ is a closed subgroup,*

(ii) *the action of G_u on the coset space $X_u = G_u/K_u$ is effective, and*

(iii) *$X'_u = G_u/Z_uK_u$ is simply connected, the natural projection $X_u \to X'_u$, is a principal torus bundle with group Z_u, and $\pi_1(X_u) \cong \pi_1(Z_u)$, free abelian of rank* $\dim Z_u$.

Proof. $\mathfrak{K}$ contains no nonzero ideal of $\mathfrak{G}$ because G is effective on X, so $\mathfrak{K}_u$ contains no nonzero ideal of $\mathfrak{G}_u$. In particular, for any choice of compact group G_u with Lie algebra $\mathfrak{G}_u$, K_u is closed in G_u and G_u acts on G_u/K_u with finite kernel.

For the unique choice decompose $\mathfrak{G}_u = \mathfrak{G}'_u + \mathfrak{Z}_u$ and let $\bar{G}_u = \bar{G}'_u \times \bar{Z}_u$ where $\bar{G}'_u$ is the compact simply connected group with Lie algebra $\mathfrak{G}'_u$. Let F be the (finite) kernel of the action of $\bar{G}_u$ on $X_u = \bar{G}_u/\bar{K}_u$ where $\bar{K}_u$ is the analytic subgroup for $\mathfrak{K}_u$. Then $G_u = \bar{G}_u/F$, $K_u = \bar{K}_u/F$, $X_u = G_u/K_u$ give us condition (ii). For (iii) note that Z_u is a torus acting freely on X_u; so we need only prove X'_u simply connected. But $X'_u = \bar{G}'_u/L$ where L is the analytic subgroup for the projection of $\mathfrak{K}_u$ to $\mathfrak{G}'_u$. This gives existence of the desired G_u; uniqueness is obvious. q.e.d.

We have constructed a "compact version" X_u of a coset space X of reductive Lie groups. Now we turn the procedure around.

Let $X = G/K$ be a coset space of compact connected Lie groups, G acting effectively. Let σ be an automorphism of $\mathfrak{G}$ such that $\sigma^2 = 1$ and $\sigma(\mathfrak{K}) = \mathfrak{K}$. Then we have (7.3a) and can define real forms of $\mathfrak{G}^C$ and $\mathfrak{K}^C$ by

$$\mathfrak{G}^* = \mathfrak{G}^\sigma + \sqrt{-1}\,\mathfrak{M}\,, \qquad \mathfrak{K}^* = \mathfrak{K}^\sigma + \sqrt{-1}\,(\mathfrak{K} \cap \mathfrak{M})\,.$$

Then $\mathfrak{G}^*$ is reductive, $\mathfrak{K}^*$ is reductive in $\mathfrak{G}^*$, and

7.5. Lemma. *There is a unique simply connected coset space $X^* = G^*/K^*$ such that* (i) *G^* is a connected Lie group with Lie algebra $\mathfrak{G}^*$,* (ii) *$\mathfrak{K}^*$ is the Lie algebra of the closed subgroup K^*, and* (iii) *G^* acts effectively on X^*.*

Let F be the torsion subgroup of $\pi_1(X)$. Then F can be viewed as a finite central subgroup of G^*_u $(=(G^*)_u)$ such that $G = G^*_u/F$, $K = (K^*_uF)/F$ and $X = X^*_u/F$.

Proof. For the first statement $G^* = \bar{G}^*/S$ and $K^* = (\bar{K}^*S)/S$ where $\bar{G}^*$ is the simply connected group for $\mathfrak{G}^*$, $\bar{K}^*$ is the analytic subgroup for $\mathfrak{K}^*$, and S is the kernel of the action of $\bar{G}^*$ on $\bar{G}^*/\bar{K}^*$. The second statement is equally transparent. q.e.d.

Lemmas 7.4 and 7.5 allow us to go back and forth between coset spaces of compact Lie groups and coset spaces of reductive Lie groups. In our applications we need only assume G reductive and then K will be a reductive subgroup. For [2, Proposition 4.1] and an obvious induction on the length of the derived series of Θ give

7.6. Lemma. *Let $\mathfrak{G}$ be a reductive Lie algebra, and Θ a solvable group consisting of automorphisms of $\mathfrak{G}$ which are fully reducible as linear transformations. Then the fixed point set $\mathfrak{G}^{\Theta}$ is a reductive subalgebra of $\mathfrak{G}$.*

To make these applications in Theorem 7.10 we need two intermediate results on invariant almost complex structures.

7.7. Proposition. *Let $X = G/K$ where G is a compact connected Lie group, K is a closed connected subgroup, and G acts effectively on X. Let σ be an involutive automorphism of G which preserves K, and thus acts on X. Let $X^* = G^*/K^*$ be the corresponding simply connected space. Extend σ to $\mathfrak{G}^C$ by complex linearity, so that σ also acts on X^*. $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$ and $\mathfrak{G}^* = \mathfrak{K}^* + \mathfrak{M}^*$ as usual.*

(i) *The G-invariant σ-invariant almost complex structures on X are in one to one correspondence with the G^*-invariant σ-invariant almost complex structures on X^*, where two structures correspond if they are equal on $\mathfrak{M}^C = \mathfrak{M}^{*C}$.*

(ii) *Suppose $\mathfrak{K}^* = \mathfrak{G}^{*\Sigma}$ where Σ is a compact subgroup of the automorphism group* Aut$(\mathfrak{G}^*)$, *suppose M^* chosen invariant under Σ, and let β denote the representation of Σ on $\mathfrak{M}^C$. If Σ induces an invariant almost complex structure on X^*,* i.e. *if $\beta = \beta' \oplus \overline{\beta'}$ with β' and $\overline{\beta'}$ disjoint, then $\mathfrak{K}^* = \mathfrak{G}^{*\Gamma}$ for some compact subgroup $\Gamma \subset$* Aut$(\mathfrak{G}^*)$ *such that Γ induces a G^*-invariant σ-invariant almost complex structure on X^*.*

Proof. Let τ (resp. τ^*) denote complex conjugation of $\mathfrak{M}^C$ over $\mathfrak{M}$ (resp. $\mathfrak{M}^*$). An invariant almost complex structure on X (resp. X^*) amounts to an ad$(\mathfrak{G}^C)$-invariant $\mathfrak{M}^C = \mathfrak{M}^+ + \mathfrak{M}^-$ where τ (resp. τ^*) interchanges $\mathfrak{M}^+$ and $\mathfrak{M}^-$. As $\sigma\tau = \tau^* = \tau\sigma$, the interchange conditions are equivalent when σ preserves $\mathfrak{M}^+$ and $\mathfrak{M}^-$, i.e. when σ preserves the almost complex structure. That proves (i).

Let A be the centralizer of $\mathfrak{K}^*$ in Aut$(\mathfrak{G}^*)$, linear algebraic group normalized by σ. Let B be a maximal compact subgroup of A normalized by σ. As $\sigma|_A$ is a Cartan involution of A, $b\sigma = \sigma b$ for all $b \in B$. Let $a \in A$ with $a\Sigma a^{-1} \subset B$. Define $\Gamma = a\Sigma a^{-1}$. Then $\mathfrak{K}^* = \mathfrak{G}^{*\Gamma}$ and (ii) follows with $\mathfrak{M}^C = \mathfrak{M}^+ + \mathfrak{M}^-$ where $a^{-1}(\mathfrak{M}^+)$ and $a^{-1}(\mathfrak{M}^-)$ are the representation spaces of β' and $\overline{\beta'}$. q.e.d.

7.8. Proposition (*cf.* [12, *Theorem* 13.3 (2)]). *Let K be a connected subgroup of maximal rank in a compact connected centerless simple Lie group G. Let α be an outer automorphism of G which preserves K, thus acts on $X = G/K$, and preserves a G-invariant almost complex structure on X. Then*

(i) $G = SU(2n)/Z_{2n}$, $K = S\{U(n) \times U(n)\}/Z_{2n}$ *and* α *interchanges the two factors* $U(n)$ *of* K*; or*

(ii) $G = SO(2n)/Z_2$, $K = \{U(n_1) \times \cdots \times U(n_s) \times SO(2m)\}/Z_2$, $n_1 + \cdots + n_s + m = n$, $m \geq 2$, *where* α *is conjugation by* $\operatorname{diag}\{P_1, \cdots, P_s; Q\}$ *with* $P_i \in U(n_i)$, $Q \in O(2m)$ *and* $\det Q = -1$*; or*

(iii) $G = E_6/Z_3$, $K = \{SU(3) \times SU(3) \times L_i\}/\{Z_3 \times Z_3\}$, $1 \leq i \leq 3$, *where* α *interchanges the two* $SU(3)$*-factors of* K, $\alpha(L_i) = L_i$, *and* $L_1 \subset L_2 \subset L_3$ *given by* $T^2 \subset S\{U(1) \times U(2)\} \subset SU(3)$.

Now we need notation for noncompact semisimple Lie groups. Compact connected simply connected groups were denoted by their Cartan classification type in boldface letters:

$$A_n = SU(n+1)\,, \quad B_n = Spin(2n+1)\,, \quad C_n = Sp(n)\,,$$
$$D_n = Spin(2n)\,, \quad G_2\,, \quad F_4\,, \quad E_6\,, \quad E_7\,, \quad E_8\,.$$

Now the complex simple simply connected groups are denoted in the obvious manner:

$$A_n^C = SL(n+1, C)\,, \quad B_n^C = Spin(2n+1, C)\,, \quad C_n^C = Sp(n, C)\,,$$
$$D_n^C = Spin(2n, C)\,, \quad G_2^C\,, \quad F_4^C\,, \quad E_6^C\,, \quad E_7^C\,, \quad E_8^C\,.$$

Further T^r denotes an r-torus, C^* denotes the multiplicative group $GL(1, C)$ of nonzero complex numbers, and we use the following standard notation on linear groups.

$S(\cdots)$: subgroup consisting of elements of determinant 1, with exceptions noted.

$O^r(n)$: real orthogonal group of $-\sum_{i=1}^{r} x_i y_i + \sum_{j=r+1}^{n} x_j y_j$.

$O(n, C)$: complex orthogonal group of $-\sum_{i=1}^{r} x_i y_i + \sum_{j=r+1}^{n} x_j y_j$.

$SO^r(n)$, $SO(n, C)$: respective identity component of $O^r(n)$ and $O(n, C)$.

$SO^*(n)$: real form of $SO(n, C)$, $n = 2m$, with maximal compact subgroup $U(m)$.

$Spin^r(n)$, $Spin(n, C)$: respective 2-sheeted (spinor contruction) covering groups of $SO^r(n)$ and $SO(n, C)$.

$U^r(n)$: complex unitary group of $-\sum_{i=1}^{r} x_i \bar{y}_i + \sum_{j=r+1}^{n} x_j \bar{y}_j$.

$Sp(n, R)$, $Sp(n, C)$: respective real and complex linear groups for the nondegenerate alternating form $\sum_{i=1}^{n} (x_i y_{i+n} - y_i x_{i+n})$ on $2n$-space.

$Sp^r(n)$: quaternion unitary group of $-\sum_{i=1}^{r} x_i \bar{y}_i + \sum_{j=r+1}^{n} x_j \bar{y}_j$.

(**7.9**) In addition we introduce the notational convention. *Centerless simple real groups are denoted with boldface for their Cartan classification type and* *the Cartan classification type of the maximal compact subgroup as a second subscript. Thus* $C_{n,A_{n-1}T^1} = Sp(n, R)/\{\pm I\}$, $B_{n,D_n} = SO^1(2n+1)$, $D_{n,A_{n-1}T^1} = SO^*(2n)/\{\pm I\}$, $C_{p+q,C_pC_q} = Sp^p(p+q)/\{\pm I\}$, *etc. The only exception is that, in expressions such as* $\{E_{7,A_7} \times T^1\}/Z_2$, *the central group being divided out (here the* Z_2*) projects monomorphically into the torus and isomorphically onto the center of the simple group. For example* $U^r(n) = \{A_{n-1,A_{r-1}A_{n-r-1}T^1} \times T^1\}/Z_n$.

Now we can describe the irreducible spaces defined by automorphisms of order 3.

7.10 Theorem. *Let* $X^* = G^*/K^*$ *be a simply connected coset space where* G^* *is a connected Lie group acting effectively. Suppose* $\mathfrak{K}^* = \mathfrak{G}^{*\theta}$ *where* θ *is an automorphism of order 3 on* $\mathfrak{G}^*$ *which does not preserve any proper ideals. Then* G^* *is reductive,* K^* *is a closed reductive subgroup, there is some number* $N \geq 2$ *of* G^**-invariant almost complex structures on* X^**, and the following tables give a complete (up to automorphism of* G^**) list of the possibilities.*

7.11. Table. G^*: centerless classical simple
K^*: centralizer of compact toral subgroup

G^*	K^*	conditions	N
$SU^m(n)/Z_n$	$S\{U^{s_1}(r_1)\times U^{s_2}(r_2)\times U^{s_3}(r_3)\}/Z_n$	$n=r_1+r_2+r_3$, $m=s_1+s_2+s_3$, $0\leq r_1\leq r_2\leq r_3$, $1\leq r_2$, $0\leq 2s_i\leq r_i$	2 if $r_1=0$; 8 if $r_1>0$
$SL(n, R)/Z_2$	$\{SL(\frac{n}{2}, C)\times T^1\}/Z_{n/2}$, $n\equiv 0(2)$		
$SL(\frac{n}{2}, Q)/Z_2$			
$SL(n, C)/Z_n$	$S\{GL(r_1, C)\times GL(r_2, C)\times GL(r_3, C)\}/Z_n$		
$SO^{2s+2t}(2n+1)$	$U^s(r)\times SO^{2t}(2n-2r+1)$	$1\leq r\leq n$, $0\leq 2s\leq r$	2 if $r=1$; 4 if $r>1$
$SO(2n+1, C)$	$GL(r, C)\times SO(2n-2r+1, C)$	$1\leq r\leq n$	
$Sp^{s+t}(n)/Z_2$	$\{U^s(r)\times Sp^t(n-r)\}/Z_2$	$1\leq r\leq n$, $0\leq 2s\leq r$, $0\leq 2t\leq n-r$	2 if $r=n$; 4 if $r<n$
$Sp(n, R)/Z_2$	$\{U^s(r)\times Sp(n-r, R)\}/Z_2$		
$Sp(n, C)/Z_2$	$\{GL(r, C)\times Sp(n-r, C)\}/Z_2$		
$SO^{2s+t}(2n)/Z_2$	$\{U^s(r)\times SO^t(2n-2r)\}/Z_2$	$1\leq r\leq n$, $0\leq 2s\leq r$, $0\leq t\leq n-r$	2 if $r=1$; 2 if $r=n$; 4 if $1<r<n$
$SO^*(2n)/Z_2$	$\{U^s(r)\times SO^*(2n-2r)\}/Z_2$		
$SO(2n, C)/Z_2$	$\{GL(r, C)\times SO(2n-2r, C)\}/Z_2$		

7.12. Table. G^*: centerless exceptional simple
K^*: centralizer of a compact toral subgroup

G^*	K^*	conditions	N
G_2	$U(2)$	—	
$G_2^*=G_{2,A_1A_1}$	$U(2)$, $U^1(2)$	—	4
G_2^C	$GL(2,C)$	—	
F_4	$\{Spin(7)\times T^1\}/Z_2$, $\{Sp(3)\times T^1\}/Z_2$	—	
F_{4,B_4}	$\{Spin^r(7)\times T^1\}/Z_2$, $\{Sp^1(3)\times T^1\}/Z_2$	$r=0,1$	4
F_{4,C_3C_1}	$\{Spin^r(7)\times T^1\}/Z_2$, $\{Sp^t(3)\times T^1\}/Z_2$ and $\{Sp(3,R)\times T^1\}/Z_2$	$r=2,3$; $t=0,1$	
F_4^C	$\{Spin(7,C)\times C^*\}/Z_2$, $\{Sp(3,C)\times C^*\}/Z_2$	—	
E_6/Z_3	$\{SO(10)\times SO(2)\}/Z_2$	—	2
	$\{S(U(5)\times U(1))\times SU(2)\}/Z_2$, $\{[SU(6)/Z_3]\times T^1\}/Z_2$	—	4
	$\{[SO(8)\times SO(2)]\times SO(2)\}/Z_2$	—	8
E_{6,A_1A_5}	$\{SO^*(10)\times SO(2)\}/Z_2$, $\{SO^4(10)\times SO(2)\}/Z_2$	—	2
	$\{S(U^r(5)\times U(1))\times SU^s(2)\}/Z_2$	$(s,r)=(0,0)$, $(0,1)$, $(0,2)$, $(1,2)$	4
	$\{[SU^r(6)/Z_3]\times T^1\}/Z_2$	$r=0,2,3$	
	$\{[SO^*(8)\times SO(2)]\times SO(2)\}/Z_2$ $\{[SO^r(8)\times SO(2)]\times SO(2)\}/Z_2$	$r=2,4$	8
E_{6,D_5T^1}	$\{SO^r(10)\times SO(2)\}/Z_2$, $\{SO^*(10)\times SO(2)\}/Z_2$	$r=0,2$	2
	$\{S(U^r(5)\times U(1))\times SU^s(2)\}/Z_2$	$(s,r)=(1,0)$, $(0,1)$, $(1,1)$, $(0,2)$	4
	$\{[SU^r(6)/Z_3]\times T^1\}/Z_2$	$r=1,2$	
	$\{[SO^*(8)\times SO(2)]\times SO(2)\}/Z_2$, $\{[SO^r(8)\times SO(2)]\times SO(2)\}/Z_2$	$r=0,2$	8
E_6^C/Z_3	$\{SO(10,C)\times C^*\}/Z_2$	—	2
	$\{S(GL(5,C)\times C^*)\times SL(2,C)\}/Z_2$ $\{[SL(6,C)/Z_3]\times C^*\}/Z_2$	—	4
	$\{[SO(8,C)\times C^*]\times C^*\}/Z_2$	—	8

G^*	K^*	conditions	N
E_7/Z_2	$\{E_6 \times T^1\}/Z_3$	—	2
	$\{SU(2)\times[SO(10)\times SO(2)]\}/Z_2$, $\{SO(2)\times SO(12)\}/Z_2$, $S(U(7)\times U(1))/Z_4$	—	4
E_{7,A_7}	$\{E_{6,A_1A_5}\times T^1\}/Z_2$	—	2
	$\{SU(2)\times[SO^*(10)\times SO(2)]\}/Z_2$ $\{SU^1(2)\times[SO^4(10)\times SO(2)]\}/Z_2$ $\{SO(2)\times SO^*(12)\}/Z_2$, $\{SO(2)\times SO^6(12)\}/Z_2$, $S(U^r(7)\times U(1))/Z_4$	$r=0, 3$	4
E_{7,A_1D_6}	$\{E_{6,D_5T^1}\times T^1\}/Z_2$, $\{E_{6,A_1A_5}\times T^1\}/Z_2$	—	2
	$\{SU^t(2)\times[SO^r(10)\times SO(2)]\}/Z_2$ $\{SU^1(2)\times[SO^*(10)\times SO(2)]\}/Z_2$ $\{SO(2)\times SO^p(12)\}/Z_2$ $S(U^s(7)\times U(1))/Z_4$	$(t, r)=(0, 0)$, $(0, 2)$, $(1, 2)$, $(0, 4)$, $p=0, 4$ $s=1, 2, 3$	4
E_{7,E_6T^1}	$\{E_6\times T^1\}/Z_3$, $\{E_{6,D_5T^1}\times T^1\}/Z_2$	—	2
	$\{SU^1(2)\times[SO(10)\times SO(2)]\}/Z_2$, $\{SU(2)\times[SO^*(10)\times SO(2)]\}/Z_2$, $\{SO(2)\times SO^*(12)\}/Z_2$, $\{SO(2)\times SO^2(12)\}/Z_2$ $S(U^r(7)\times U(1))/Z_4$	$r=1, 2$	4
E_7^C/Z_2	$\{E_6^C\times C^*\}Z_3$	—	2
	$\{SL(2, C)\times[SO(10, C)\times C^*]\}/Z_2$, $\{C^*\times SO(12, C)\}/Z_2$, $S\{GL(7, C)\times C^*\}/Z_4$	—	4
E_8	$SO(14)\times SO(2)$, $\{E_7\times T^1\}/Z_2$	—	4
E_{8,D_8}	$SO(14)\times SO(2)$, $SO^6(14)\times SO(2)$, $SO^*(14)\times SO(2)$, $\{E_{7,A_1D_6}\times T^1\}/Z_2$, $\{E_{7,A_7}\times T^1\}/Z_2$	—	
E_{8,A_1E_7}	$SO^2(14)\times SO(2)$, $SO^4(14)\times SO(2)$, $SO^*(14)\times SO(2)$, $\{E_7\times T^1\}/Z_2$, $\{E_{7,E_6T^1}\times T^1\}/Z_2$, $\{E_{7,A_1D_6}\times T^1\}/Z_2$	—	
E_8^C	$SO(14, C)\times C^*$, $\{E_7^C\times C^*\}/Z_2$	—	

7.13. Table. G^*: centerless simple, rank G^*=rank K^*
K^*: not the centralizer of a torus
{$N=2$, G^* is exceptional, and K^* has center of order 3.}

G^*	K^*
G_2	$SU(3)$
$G_2^*=G_{2,A_1A_1}$	$SU^1(3)$
G_2^C	$SL(3,C)$
F_4	$\{SU(3)\times SU(3)\}/Z_3$
F_{4,B_4}	$\{SU^1(3)\times SU(3)\}/Z_3$
F_{4,C_3C_1}	$\{SU(3)\times SU^1(3)\}/Z_3$, $\{SU^1(3)\times SU^1(3)\}/Z_3$
F_4^C	$\{SL(3,C)\times SL(3,C)\}/Z_3$
E_6/Z_3	$\{SU(3)\times SU(3)\times SU(3)\}/\{Z_3\times Z_3\}$
E_{6,A_1A_5}	$\{SU^1(3)\times SU(3)\times SU(3)\}/\{Z_3\times Z_3\}$ $\{SU^1(3)\times SU^1(3)\times SU^1(3)\}/\{Z_3\times Z_3\}$
E_{6,D_5T^1}	$\{SU^1(3)\times SU^1(3)\times SU(3)\}/\{Z_3\times Z_3\}$
E_{6,F_4}	$\{SL(3,C)\times SU(3)\}/Z_3$
E_{6,C_4}	$\{SL(3,C)\times SU^1(3)\}/Z_3$
E_6^C/Z_3	$\{SL(3,C)\times SL(3,C)\times SL(3,C)\}/\{Z_3\times Z_3\}$
E_7/Z_2	$\{SU(3)\times[SU(6)/Z_2]\}/Z_3$
E_{7,A_7}	$\{SU(3)\times[SU^1(6)/Z_2]\}/Z_3$, $\{SU^1(3)\times[SU^3(6)/Z_2]\}/Z_3$
E_{7,A_1D_6}	$\{SU^1(3)\times[SU(6)/Z_2]\}/Z_3$, $\{SU(3)\times[SU^2(6)/Z_2]\}/Z_3$, $\{SU^1(3)\times[SU^2(6)/Z_2]\}/Z_3$
E_{7,E_6T^1}	$\{SU^1(3)\times[SU^1(6)/Z_2]\}/Z_3$, $\{SU(3)\times[SU^3(6)/Z_2]\}/Z_3$
E_7^C	$\{SL(3,C)\times[SL(6,C)/Z_2]\}/Z_3$
E_8	$\{SU(3)\times E_6\}/Z_3$, $SU(9)/Z_3$
E_{6,D_8}	$\{SU(3)\times E_{6,D_5T^1}\}/Z_3$, $\{SU^1(3)\times E_{6,A_1A_5}\}Z_3$ $SU^1(9)/Z_3$, $SU^4(9)/Z_3$
E_{8,A_1E_7}	$\{SU^1(3)\times E_6\}/Z_3$, $\{SU^1(3)\times E_{6,D_5T^1}\}/Z_3$, $\{SU(3)\times E_{6,A_1A_5}\}/Z_3$, $SU^2(9)/Z_3$, $SU^3(9)/Z_3$
E_8^C	$\{SL(3,C)\times E_6^C\}/Z_3$, $SL(9,C)/Z_3$

7.14. Table. Rank $G^* >$ rank K^*

G^*	K^*	conditions	N
$Spin(8)$	$SU(3)/Z_3$	—	2
$SO^4(8)$	$SU^1(3)/Z_3$	—	
$Spin(8, C)$	$SL(3, C)/Z_3$	—	
$Spin(8)$, $Spin^1(8)$	G_2	—	note (1)
$Spin^3(8)$, $Spin^4(8)$	G_2^*	—	
$Spin(8, C)$	G_2^C	—	note (2)
$\{L^* \times L^* \times L^*\}/\delta Z^*$	$\delta L^*/\delta Z^*$	note (3)	note (1)
$\{L^C \times L^*\}/\dot{\delta} Z^*$	$\bar{\delta} L^*/\delta Z^*$	note (4)	
$\{L^C \times L^C \times L^C\}/\delta Z$	$\delta L^C/\delta Z$	note (3)	note (2)
vector group R^2	$\{0\}$	—	note (1)

note 1	one-one correspondence with 2×2 real matrices of square -I
note 2	one-one correspondence with 2×2 complex matrices of square -I
note 3	$\mathfrak{L}$ is an arbitrary compact simple Lie algebra. $\mathfrak{L}^*$ is an arbitrary real form of $\mathfrak{L}^C$. L^* and L^C denote the connected simply connected Lie groups with Lie algebras $\mathfrak{L}^*$ and $\mathfrak{L}^C$; Z^* and Z denote their centers. $\delta(x)$ denotes (x, x, x).
note 4	$\bar{\delta}(x)$ denotes $(\pi(x), x)$ where $\pi: L^* \to L^C$ gives the universal covering of the R-analytic subgroup of L^C with Lie algebra $\mathfrak{L}^*$.

Proof. If $\mathfrak{G}^*$ is not semisimple then it has radical $\mathfrak{R} \neq 0$. Let $\mathfrak{S}$ be the last nonzero term of the derived series of $\mathfrak{R}$. Then $\mathfrak{S}$ is an abelian Lie subalgebra stable under θ. Now $\mathfrak{G}^* = \mathfrak{S}$ and $\dim \mathfrak{S} = 2$ because $\mathfrak{G}^*$ has no proper θ-invariant ideal. Thus G^* is a 2-dimentional vector group, $K^* = \{0\}$ and $X^* = R^2$.

If $\mathfrak{G}^*$ is not semisimple we have just seen that it is abelian. So $\mathfrak{G}^*$ is reductive, and now $\mathfrak{K}^* = \mathfrak{G}^{*\Theta}$ $[\Theta = \{1, \theta, \theta^2\}]$ is a reductive subalgebra by Lemma 7.6. In particular we have a θ-stable $ad(K^*)$-stable decomposition $\mathfrak{G}^* = \mathfrak{K}^* + \mathfrak{M}^*$ and $\theta|_{\mathfrak{M}^*} = \cos \frac{2\pi}{3} I \pm \sin \frac{2\pi}{3} J$ defines two G^*-invariant almost complex structures on X^*.

We may now assume $\mathfrak{G}^*$ semisimple. Extend θ by linearity to an automorphism of $\mathfrak{G}^{*C}$ and let B be a maximal compact subgroup of Aut$(\mathfrak{G}^{*C})$ containing θ. B specifies a compact real form $\mathfrak{G}$ of $\mathfrak{G}^{*C}$ by: $\exp(ad\mathfrak{G})$ is the identity component B_0. Now $\theta(\mathfrak{G}) = \mathfrak{G}$. Let $\mathfrak{K} = \mathfrak{G}^\theta$ and let $X = G/K$ be the simply connected coset space, G connected and acting effectively on X, defined by $(\mathfrak{G}, \mathfrak{K})$. Let $X_u^* = G_u^*/K_u^*$ as in Lemma 7.4. There is an automorphism α of $\mathfrak{G}^{*C}$ sending $\mathfrak{G}$ to $\mathfrak{G}_u^*$. As $\alpha(\mathfrak{K}) \cong \mathfrak{K} \cong \mathfrak{K}_u^*$ because the latter two are compact real forms of $(\mathfrak{G}^{*C})^\theta$ there is an automorphism β of $\mathfrak{G}_u^* = \alpha(\mathfrak{G})$ sending $\alpha(\mathfrak{K})$ to $\mathfrak{K}_u^*$. Now $\beta\alpha : \mathfrak{G} \cong \mathfrak{G}_u^*$ sending $\mathfrak{K}$ to $\mathfrak{K}_u^*$, X to X_u^*. Thus we may view $X^* = G^*/K^*$ as constructed from $X = G/K$ as in Lemma 7.5, provided that we view $\mathfrak{K}$ as $\mathfrak{G}^\varphi$ where $\varphi = \beta\alpha\theta\alpha^{-1}\beta^{-1}$. In other words, we are in the duality of Lemmas 7.4 and 7.5, except that $\mathfrak{K} = \mathfrak{G}^\varphi$ and $\mathfrak{K}^* = \mathfrak{G}^{*\theta}$ where the only relation between φ and θ is their conjugacy in Aut$(\mathfrak{G}^C)$. In particular, if σ is the Cartan involution of $\mathfrak{G}^*$ preserving $\mathfrak{K}^*$, hence the involutive automorphism which defines $X^* = G^*/K^*$ from $X = G/K$, σ need not commute with θ nor with φ.

We apply the hypothesis that $\mathfrak{G}^*$ has no proper ideal preserved by θ. As θ has order 3 it says that there are just four cases, as follows[4].

1. $\mathfrak{G}^C$ is simple.
2. $\mathfrak{G}^*$ is simple but $\mathfrak{G}^C$ is not.
3. $\mathfrak{G}^* = \mathfrak{L}^* \oplus \mathfrak{L}^* \oplus \mathfrak{L}^*$ with $\mathfrak{L}^{*C}$ simple.
4. $\mathfrak{G}^* = \mathfrak{L}^* \oplus \mathfrak{L}^* \oplus \mathfrak{L}^*$ with $\mathfrak{L}^*$ simple, $\mathfrak{L}^{*C}$ not simple.

In cases 3 and 4, θ acts by cyclic permutation of the summands $\mathfrak{L}^*$.

In case 2, $\mathfrak{G}^C = \mathfrak{H} \oplus \mathfrak{H}$ where $\mathfrak{H}$ is a complex simple Lie algebra, $\mathfrak{G}^*$ is isomorphic to $\mathfrak{H}$ as a real Lie algebra, and $\mathfrak{G}^*$ is embedded diagonally. θ extends to $\mathfrak{G}^C$ as $\psi \times \psi$ where ψ has order 3. Now $\mathfrak{G} = \mathfrak{L} \oplus \mathfrak{L}$ where $\mathfrak{L}$ is a compact real form of $\mathfrak{H}$, and $\varphi = \nu \times \nu$ where ν has order 3 on $\mathfrak{L}$. Thus $X = G/K$ is given as $(L \times L)/(S \times S) = (L/S) \times (L/S)$ where L is simple and L/S is listed in Theorem 6.1, while $X^* = L^C/S^C$.

In case 4, the same arguments show $X = G/K$ to be $(A \times A)/(B \times B) = (A/B) \times (A/B)$ where A/B is the space listed in Theorem 6.1 with A not simple, and $X^* = G^*/K^*$ is A^C/B^C.

In cases 1 and 3, $X = G/K$ is listed in Theorem 6.1. We go on to consider those cases.

We first consider the case where σ is inner on G. If rank $G =$ rank K (tables 1 and 2 of Theorem 6.1) then Propositions 6.4 and 7.7 say $\sigma = ad(k)$ for some $k \in K$. Note k^2 central in G because $\sigma^2 = 1$. Now we run through the list.

$\boldsymbol{SU(n)/S\{U(r_1) \times U(r_2) \times U(r_3)\}}$. We may conjugate in K and assume k diagonal. k^2 is scalar so k has just two eigenvalues. Now $G^* = SU^m(n)$ and

[4] If we had θ of order k, and m were the number of divisors $d \geq 1$ of k, then we would have 2^m cases in the obvious manner.

$K^* = S\{U^{s_1}(r_1) \times U^{s_2}(r_2) \times U^{s_3}(r_3)\}$, $m = s_1 + s_2 + s_3$, with normalization $2s_i \leq r_i$.

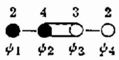
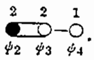
$SO(2n+1)/U(r) \times SO(2n-2r+1)$. Here $k^2 = I$ so k is of form $\mathrm{diag}\{-I_s, I_{r-s}\} \times \mathrm{diag}\{-I_{2t}, I_{2n-2r-2t+1}\}$. Now $G^* = SO^{2(s+t)}(2n+1)$ and $K^* = U^s(r) \times SO^{2t}(2n-2r+1)$ with the normalization $s \leq r/2$.

$Sp(n)/U(r) \times Sp(n-r)$. Here $k^2 = \pm I$ and may be assumed diagonal. If $k^2 = -I$ then $G^* = Sp(n, R)$ and $K^* = U^s(r) \times Sp(n-r, R)$, $s \leq r/2$. If $k^2 = I$ then $G^* = Sp^{s+t}(n)$ and $K^* = U^s(r) \times Sp^t(n-r)$, $s \leq r/2, t \leq \frac{1}{2}(n-r)$.

$SO(2n)/U(r) \times SO(2n-2r)$. Here $k^2 = \pm I$. If $k^2 = -I$ then[5] $G^* = SO^*(2n)$ and $K^* = U^s(r) \times SO^*(2n-2r)$, $s \leq r/2$. If $k^2 = I$ then $G^* = SO^{2s+2t}(2n)$ and $K^* = U^s(r) \times SO^{2t}(2n-2r)$, $s \leq r/2$, $2t \leq n-r$.

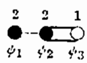
$G_2/U(2)$. If $k = 1$ then $G^* = G_2$ and $K^* = U(2)$. If $k \neq 1$ but k is central in K, then $G^* = G_2^*$ (unique noncompact form of G_2, equal to G_{2, A_1A_1}) and $K^* = U(2)$. If k is not central in K then $G^* = G_2^*$ and $K^* = U^1(2)$.

If $G = F_4$ then either (i) $\sigma = 1$, or (ii) $G^* = F_{4, B_4}$ with $\dim \mathfrak{G}^\sigma = 36$, or (iii) $G^* = F_{4, C_3C_1}$ with $\dim \mathfrak{G}^\sigma = 24$.

$F_4/Spin(7) \cdot T^1$. Here G has diagram  and the semisimple part K' of K has diagram . Now the vertices modulo $v_1 R$ of the fundamental simplex of K' are 0, $v_2' = 2v_2$, $v_3' = \frac{3}{2}v_3$ and $v_4' = 2v_4$. Thus we may restrict attention to $k = \exp(2\pi\sqrt{-1}\,x)$ where[6]

x	O	v_1	$2v_2$	$2v_2+v_1$	$\frac{3}{2}v_3$	$\frac{3}{2}v_3+v_1$	v_4	v_4+v_1
G^*	F_4	F_{4,B_4}	F_{4,B_4}	F_{4,B_4}	F_{4,C_3C_1}	F_{4,C_3C_1}	F_{4,C_3C_1}	F_{4,C_3C_1}
K^*	B_3T^1		$B_{3,D_3}T^1$		$B_{3,B_1D_2}T^1$		$B_{3,B_2D_1}T^1$	

where $D_3 = A_3$, $B_1 = A_1$, $D_2 = A_1 \oplus A_1$ and $D_1 = T^1$.

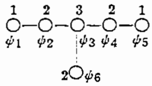
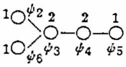
$F_4/Sp(3) \cdot T^1$. K' has diagram  and $v_1' = v_1 \sim 2v_2 = v_2'$ in K'. Now as above, we need only note

x	O	v_4	v_1	v_1+v_4	$\frac{3}{2}v_3$	$\frac{3}{2}v_3+v_4$
G^*	F_4	F_{4,C_3C_1}	F_{4,B_4}	F_{4,C_3C_1}	F_{4,C_3C_1}	F_{4,C_3C_1}
K^*	C_3T^1		$C_{3,C_1C_2}T^1$		$C_{3,A_2T^1}T^1$	

If $G = E_6$ then either (i) $\sigma = 1$, or (ii) $G^* = E_{6, A_1A_5}$ with $\dim \mathfrak{G}^\sigma = 38$, or (iii) $G^* = E_{6, D_5T^1}$ with $\dim \mathfrak{G}^\sigma = 46$.

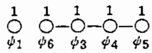
[5] $SO^*(2m)$ is the noncompact form with maximal compact subgroup $U(m)$.

[6] Determination of K^* is obvious, of G^* is obtained by counting roots with integer values on x.

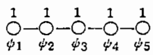
$E_6/SO(10)\cdot SO(2)$. Here G has diagram  and K' has diagram . As $\frac{1}{2}v_2' \sim \frac{1}{2}v_6'$ and $v_3' \sim v_4'$ in K', we need only note

x	O	$\frac{1}{2}v_1$	v_2	$v_2+\frac{1}{2}v_1$	$\frac{3}{2}v_3$	$\frac{3}{2}v_3+\frac{1}{2}v_1$	$\frac{1}{2}v_5$	$\frac{1}{2}v_5+\frac{1}{2}v_1$
G^*	E_6	E_{6,D_5T^1}	E_{6,A_1A_5}	E_{6,D_5T^1}	E_{6,A_1A_5}		E_{6,D_5T^1}	
K^*	D_5T^1		$D_{5,A_4T^1}\cdot T^1$		$D_{5,A_1A_1A_3}\cdot T^1$		$D_{5,D_4T^1}\cdot T^1$	

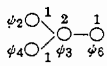
were $D_{5,A_1A_1A_3} = D_{5,D_2D_3}$.

$E_6/S(U(5)\cdot U(1))\cdot SU(2)$. K' has diagram  and $v_6' \sim v_5'$, $v_3' \sim v_4'$ in K'. Now we need only check

x	O	v_2	$\frac{1}{2}v_1$	$\frac{1}{2}v_1+v_2$	v_6	v_6+v_2
G^*	E_6	E_{6,A_1A_5}	E_{6,D_5T^1}	E_{6,D_5T^1}	E_{6,A_1A_5}	E_{6,D_5T^1}
K^*	$A_1A_4T^1$		$A_{1,T^1}A_4T^1$		$A_1A_{4,A_3T^1}T^1$	
x	$v_6+\frac{1}{2}v_1$	$v_6+\frac{1}{2}v_1+v_2$	$\frac{3}{2}v_3$	$\frac{3}{2}v_3+v_2$	$\frac{3}{2}v_3+\frac{1}{2}v_1$	$\frac{3}{2}v_3+\frac{1}{2}v_1+v_2$
G^*	E_{6,D_5T^1}	E_{6,D_5T^1}	E_{6,A_1A_5}	E_{6,D_5T^1}	E_{6,A_5A_1}	E_{6,A_5A_1}
K^*	$A_{1,T^1}A_{4,A_3T^1}T^1$		$A_1A_{4,A_1A_2T^1}T^1$		$A_{1,T^1}A_{4,A_1A_2T^1}T^1$	

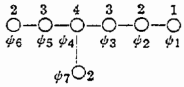
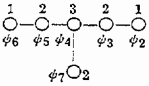
$E_6/U(6)$. K' has diagram  and $v_1' \sim v_5'$, $v_2' \sim v_4'$ in K'. Thus we need only check

x	O	v_6	$\frac{1}{2}v_1$	$\frac{1}{2}v_1+v_6$	v_2	v_2+v_6	$\frac{3}{2}v_3$	$\frac{3}{2}v_3+v_6$
G^*	E_6	E_{6,A_1A_5}	E_{6,D_5T^1}	E_{6,D_5T^1}	E_{6,A_1A_5}	E_{6,D_5T^1}	E_{6,A_1A_5}	E_{6,A_1A_5}
K^*	A_5T^1		$A_{5,A_4T^1}T^1$		$A_{5,A_1A_3T^1}T^1$		$A_{5,A_2A_2T^1}T^1$	

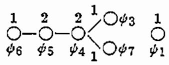
$E_6/SO(8)\cdot SO(2)\cdot SO(2)$. K' has diagram  with $\frac{1}{2}v_2' \sim \frac{1}{2}v_4' \sim \frac{1}{2}v_6'$ in K'. Now we need only check.

x	O	$\frac{1}{2}v_1$	$\frac{1}{2}v_5$	$\frac{1}{2}(v_1+v_5)$	v_2	$v_2+\frac{1}{2}v_1$
G^*	E_6	E_{6,D_5T^1}	E_{6,D_5T^1}	E_{6,D_5T^1}	E_{6,A_1A_5}	E_{6,D_5T^1}
K^*	$D_4\cdot T^2$				$D_{4,A_3T^1}\cdot T^2$	
x	$v_2+\frac{1}{2}v_5$	$v_2+\frac{1}{2}(v_1+v_5)$	$\frac{3}{2}v_3$	$\frac{3}{2}v_3+\frac{1}{2}v_1$	$\frac{3}{2}v_3+\frac{1}{2}v_5$	$\frac{3}{2}v_3+\frac{1}{2}(v_1+v_5)$
G^*	E_{6,A_1A_5}	E_{6,D_5T^1}	E_{6,A_1A_5}	E_{6,A_1A_5}	E_{6,A_1A_5}	E_{6,A_1A_5}
K^*	$D_{4,A_3T^1}\cdot T^2$		$D_{4,A_1A_1A_1A_1}\cdot T^2$			

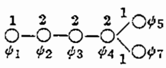
If $G = E_7$ then (i) $\sigma = 1$ or (ii) $G^* = E_{7,A_7}$ with $\dim \mathfrak{G}^\sigma = 63$, or (iii) $G^* = E_{7,A_1D_6}$ with $\dim \mathfrak{G}^\sigma = 69$, or (iv) $G^* = E_{7,E_6T^1}$ with $\dim \mathfrak{G}^\sigma = 79$.

E_7/E_6T^1. Here G has diagram  and K' has diagram . As $v_5' \sim v_3' \sim v_7'$ and $\frac{1}{2}v_2' \sim \frac{1}{2}v_6'$ in K', we need only check

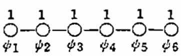
x	O	$\frac{1}{2}v_1$	v_2	$v_2+\frac{1}{2}v_1$	$\frac{3}{2}v_3$	$\frac{3}{2}v_3+\frac{1}{2}v_1$
G^*	E_7	E_{7,E_6T^1}	E_{7,A_1D_6}	E_{7,E_6T^1}	E_{7,A_7}	E_{7,A_1D_6}
K^*	E_6T^1		$E_{6,D_5T^1}\cdot T^1$		$E_{6,A_1A_5}\cdot T^1$	

$E_7/SU(2)\cdot SO(10)\cdot SO(2)$. K' has diagram , $\frac{1}{2}v_3' \sim \frac{1}{2}v_7'$ and $v_4' \sim v_5'$. Thus we need only check

x	O	v_2	$\frac{1}{2}v_1$	$\frac{1}{2}v_1+v_2$	$\frac{3}{2}v_3$	$\frac{3}{2}v_3+v_2$	$\frac{3}{2}v_3+\frac{1}{2}v_1$	$\frac{3}{2}v_3+\frac{1}{2}v_1+v_2$
G^*	E_7	E_{7,A_1D_6}	E_{7,E_6T^1}		E_{7,A_7}	E_{7,E_6T^1}	E_{7,A_1D_6}	E_{7,A_1D_6}
K^*	$D_5A_1T^1$		$D_5A_{1,T^1}T^1$		$D_{5,A_4T^1}A_1T^1$		$D_{5,A_4T^1}A_{1,T^1}T^1$	
x	$2v_4$	$2v_4+v_2$	$2v_4+\frac{1}{2}v_1$	$2v_4+\frac{1}{2}v_1+v_2$	v_6	v_6+v_2	$v_6+\frac{1}{2}v_1$	$v_6+\frac{1}{2}v_1+v_2$
G^*	E_{7,A_1D_6}		E_{7,A_7}		E_{7,A_1D_6}		E_{7,A_1D_6}	
K^*	$D_{5,A_1A_1A_3}A_1T^1$		$D_{5,A_1A_1A_3}A_{1,T^1}T^1$		$D_{5,D_4T^1}A_1T^1$		$D_{5,D_4T^1}A_{1,T^1}T^1$	

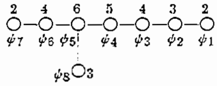
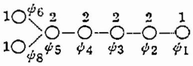
$E_7/SO(12)\times SO(2)$. Here K':  with $\frac{1}{2}v_5' \sim \frac{1}{2}v_7'$ and $v_2' \sim v_4'$. So we need only observe

x	O	v_6	$\frac{1}{2}v_1$	$\frac{1}{2}v_1+v_6$	v_2	v_2+v_6	$\frac{3}{2}v_3$	$\frac{3}{2}v_3+v_6$	v_7	v_7+v_6
G^*	E_7	E_{7,A_1D_6}	E_{7,E_6T^1}		E_{7,A_1D_6}		E_{7,A_7}		E_{7,A_7}	E_{7,E_6T^1}
K^*	D_6T^1		$D_{6,D_5T^1}T^1$		$D_{6,A_1A_1D_4}T^1$		$D_{6,A_3A_3}T^1$		$D_{6,A_5T^1}T^1$	

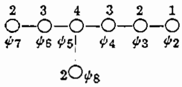
$E_7/U(7)$. Here K':  with $\frac{1}{2}v_1' \sim \frac{1}{2}v_6'$, $\frac{1}{2}v_2' \sim \frac{1}{2}v_5'$ and $\frac{1}{2}v_3' \sim \frac{1}{2}v_4'$. Now we need only check

x	O	v_7	$\frac{1}{2}v_1$	$\frac{1}{2}v_1+v_7$	v_2	v_2+v_7	$\frac{3}{2}v_3$	$\frac{3}{2}v_3+v_7$
G^*	E_7	E_{7,A_7}	E_{7,E_6T^1}	E_{7,A_1D_6}	E_{7,A_1D_6}	E_{7,E_6T^1}	E_{7,A_7}	E_{7,A_1D_6}
K^*	A_6T^1		$A_{6,A_5T^1}\cdot T^1$		$A_{6,A_1A_4T^1}T^1$		$A_{6,A_2A_3T^1}\cdot T^1$	

If $G = \boldsymbol{E}_8$ then (i) $\sigma = 1$, or (ii) $G^* = \boldsymbol{E}_{8,D_8}$ with $\dim \mathfrak{G}^\sigma = 120$, or (iii) $G^* = \boldsymbol{E}_{8,A_1E_7}$ with $\dim \mathfrak{G}^\sigma = 136$.

$\boldsymbol{E}_8/\boldsymbol{SO}(14)\cdot\boldsymbol{SO}(2)$. G has diagram  and K' has diagram . As $\frac{1}{2}v_6' \sim \frac{1}{2}v_8'$, $v_2' \sim v_5'$ and $v_3' \sim v_4'$ we now need only observe

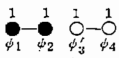
x	O	v_7	v_1	v_1+v_7	$\frac{3}{2}v_2$	$\frac{3}{2}v_2+v_7$	$2v_3$	$2v_3+v_7$	$\frac{3}{2}v_8$	$\frac{3}{2}v_8+v_7$
G^*	E_8	E_{8,D_8}	E_{8,A_1E_7}		E_{8,A_1E_7}		E_{8,D_8}		E_{8,D_8}	E_{8,A_1E_7}
K^*	D_7T^1		$D_{7,D_6T^1}T^1$		$D_{7,A_1A_1D_5}T^1$		$D_{7,A_3D_4}T^1$		$D_{7,A_6T^1}T^1$	

$\boldsymbol{E}_8/\boldsymbol{E}_7\cdot\boldsymbol{T}^1$. K' has diagram  with $v_3' \sim v_7'$. Now we need only check

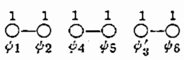
x	O	v_1	$\frac{3}{2}v_2$	$\frac{3}{2}v_2+v_1$	$2v_3$	$2v_3+v_1$	$\frac{3}{2}v_8$	$\frac{3}{2}v_8+v_1$
G^*	E_8	E_{8,A_1E_7}	E_{8,A_1E_7}	E_{8,A_1E_7}	E_{8,D_8}	E_{8,A_1E_7}	E_{8,D_8}	E_{8,D_8}
K^*	$E_7\cdot T^1$		$E_{7,E_6T^1}T^1$		$E_{7,A_1D_6}\cdot T^1$		$E_{7,A_7}\cdot T^1$	

This completes our run through table 1 of Theorem 6.1 for σ inner. We go on to table 2.

$\boldsymbol{G}_2/\boldsymbol{SU}(3)$. If $\sigma = 1$ then $G^* = \boldsymbol{G}_2$ and $K^* = \boldsymbol{SU}(3)$. If $\sigma \neq 1$ then $G^* = \boldsymbol{G}_2^*$ and $K^* = \boldsymbol{SU}^1(3)$, for those are the only possibilities.

$\boldsymbol{F}_4/\boldsymbol{A}_2\boldsymbol{A}_2$. K has diagram  where $\psi_3' = -(2\psi_1 + 4\psi_2 + 3\psi_3 + 2\psi_4)$, so the vertices of its fundamental simplex are $v_0' = 0$, $v_1' = 2v_1 - 2v_3$, $v_2' = 4v_2 - 4v_3$, $v_3' = -v_3$ and $v_4' = 2v_4 - 2v_3$. As $\frac{1}{2}v_1' \sim \frac{1}{2}v_2'$ and $\frac{1}{2}v_3' \sim \frac{1}{2}v_4'$ by an inner automorphism of G which preserves K, now we need only calculate $\dim \mathfrak{G}^\sigma$, $\sigma = ad(k)$, $k = \exp 2\pi\sqrt{-1}\, x \in K$, as follows.

x	O	$\frac{3}{2}v_1' \sim v_1$	$\frac{3}{2}v_3' \sim \frac{3}{2}v_3$	$\frac{3}{2}v_1' + \frac{3}{2}v_3' \sim v_1 - \frac{3}{2}v_3$
G^*	F_4	F_{4,B_4}	F_{4,C_3C_1}	F_{4,C_3C_1}
K^*	A_2A_2	$A_{2,A_1T^1}A_2$	A_2A_{2,A_1T^1}	$A_{2,A_1T^1}\cdot A_{2,A_1T^1}$

$\boldsymbol{E}_6/\boldsymbol{A}_2\boldsymbol{A}_2\boldsymbol{A}_2$. K has diagram  with $\psi_3' = -(\psi_1 + 2\psi_2 + 3\psi_3 + 2\psi_4 + \psi_5 + 2\psi_6)$; its fundamental simplex has vertices $v_0' = 0$, $v_1' = v_1 - v_3$, $v_2' = 2v_2 - 2v_3$, $v_3' = -v_3$, $v_4' = 2v_4 - 2v_3$, $v_5' = v_5 - v_3$ and $v_6' = 2v_6 - 2v_3$.

An automorphism of G preserves K and permutes its summands cyclically, inducing

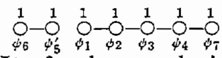
$$v_0' \to v_0' , \quad v_1' \to v_3' \to v_5' \to v_1' , \quad v_2' \to v_6' \to v_4' \to v_2' .$$

Another acts by

$$v_0' \to v_0' , \quad v_3' \to v_6' \to v_3' , \quad v_1' \to v_4' \to v_1' , \quad v_2' \to v_5' \to v_2' ,$$

given by $w\alpha$, $w \in W$ with $w\alpha\mathfrak{D}_0 = \mathfrak{D}_0$, α given on a maximal torus of K by $t \to t^{-1}$. Now we need only calculate dim $\mathfrak{G}^\sigma$ in the cases

x	O	$\frac{3}{2}v_1' \sim \frac{1}{2}(v_1+3v_3)$	$\frac{3}{2}(v_1'+v_3') \sim \frac{1}{2}v_1$ and $\frac{1}{2}(v_1'+v_6') \sim \frac{1}{2}(v_1+3v_3+2v_6)$	$\frac{1}{2}(v_1'+v_3'+v_5') \sim \frac{1}{2}(v_1+3v_3+v_5)$ $\frac{3}{2}(v_1'+v_3'+v_4') \sim \frac{1}{2}v_1+v_4$ $\frac{3}{2}(v_1'+v_4'+v_6') \sim \frac{1}{2}v_1+v_4+v_6$
G^*	E_6	E_{6,A_1A_5}	E_{6,D_5T^1}	E_{6,A_1A_5}
K^*	$A_2 \cdot A_2 \cdot A_2$	$A_{2,A_1T^1} \cdot A_2 \cdot A_2$	$A_{2,A_1T^1} \cdot A_{2,A_1T^1} \cdot A_2$	$A_{2,A_1T^1} \cdot A_{2,A_1T^1} \cdot A_{2,A_1T^1}$

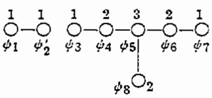
E_7/A_2A_5. K has diagram 

with $\phi_5' = -(\phi_1 + 2\phi_2 + 3\phi_3 + 4\phi_4 + 3\phi_5 + 2\phi_6 + 2\phi_7)$. Its fundamental simplex has vertices $v_0' = 0$, $v_1' = v_1 - v_5$, $v_2' = 2v_2 - 2v_5$, $v_3' = 3v_3 - 3v_5$, $v_4' = 4v_4 - 4v_5$, $v_5' = -v_5$, $v_6' = 2v_6 - 2v_5$ and $v_7' = 2v_7 - 2v_5$. The center is generated by an element conjugate to its inverse, and that conjugation gives

$$v_0' \leftrightarrow v_0', \quad v_5' \leftrightarrow v_6', \quad v_1' \leftrightarrow v_7', \quad v_2' \leftrightarrow v_4', \quad v_3' \leftrightarrow v_3' .$$

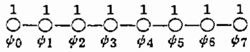
Now we need only check the following determinations of $\mathfrak{G}^\sigma$.

x	O	$\frac{3}{2}v_6'$	$\frac{3}{2}v_1'$	$\frac{3}{2}(v_1'+v_6')$ $\frac{3}{2}(v_1'+v_5')$
G^*	E_7	E_{7,A_1D_6}	E_{7,A_7}	E_{7,E_6T^1}
K^*	A_2A_5	$A_{2,A_1T^1}A_5$	A_2A_{5,A_4T^1}	$A_{2,A_1T^1}A_{5,A_4T^1}$
x	$\frac{3}{2}v_2'$	$\frac{3}{2}(v_2'+v_6')$ $\frac{3}{2}(v_2'+v_5')$	$\frac{3}{2}v_3'$	$\frac{3}{2}(v_3'+v_6')$
G^*	E_{7,A_1D_6}	E_{7,A_1D_6}	E_{7,E_6T^1}	E_{7,A_7}
K^*	$A_2A_{5,A_1A_3T^1}$	$A_{2,A_1T^1}A_{5,A_1A_3T^1}$	$A_2A_{5,A_2A_2T^1}$	$A_{2,A_1T^1}A_{5,A_2A_2T^1}$

E_8/A_2E_6. K has diagram 

with $\phi_2' = -(2\phi_1 + 3\phi_2 + 4\phi_3 + 5\phi_4 + 6\phi_5 + 4\phi_6 + 2\phi_7 + 3\phi_8)$. As above we now need only check

x	O	$\frac{3}{2}v_1'$	$\frac{3}{2}v_3'$	$\frac{3}{2}(v_3'+v_1')$ $\frac{3}{2}(v_3'+v_2')$	$3v_4'$	$3v_4'+\frac{3}{2}v_1'$ $3v_4'+\frac{3}{2}v_2'$
G	E_8	E_{8,A_1E_7}	E_{8,D_8}	E_{8,A_1E_7}	E_{8,A_1E_7}	E_{8,D_8}
K^*	A_2E_6	$A_{2,A_1T^1}E_6$	A_2E_{6,D_5T^1}	$A_{2,A_1T^1}E_{6,D_5T^1}$	A_2E_{6,A_1A_5}	$A_{2,A_1T^1}E_{6,A_1A_5}$

E_8/A_8. K has diagram 

with $\phi_0 = -(2\phi_1 + 3\phi_2 + 4\phi_3 + 5\phi_4 + 6\phi_5 + 4\phi_6 + 2\phi_7 + 3\phi_8)$. Now $v_0' = -v_8$, $v_1' = 2v_1 - 2v_8$, $v_2' = 3v_2 - 3v_8$, $v_3' = 4v_3 - 4v_8$, and we need only check the cases

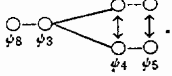
x	O	$\frac{3}{2}v_0'$	$\frac{3}{2}v_1'$	$\frac{3}{2}v_2'$	$\frac{3}{2}v_3'$
G^*	E_8	E_{8,D_8}	E_{8,A_1E_7}	E_{8,A_1E_7}	E_{8,D_8}
K^*	A_8	A_{8,A_7T^1}	$A_{8,A_1A_6T^1}$	$A_{8,A_2A_5T^1}$	$A_{8,A_3A_4T^1}$

This completes our run through tables 1 and 2 of Theorem 6.1 for σ inner. If σ is outer there, then Proposition 7.8 says that (a) $G = E_6/Z_3$ and $K = A_2 \cdot A_2 \cdot A_2$ with σ interchanging the first two factors and preserving the third, or (b) $G = SU(n)/Z_n$ with $n = 2r$ and $K = S\{U(r) \times U(r)\}$, σ interchanging the two factors of K, or (c) $G = SO(2n)/Z_2$ and $K = \{U(r) \times SO(2n-2r)\}/Z_2$ with $1 \leq r \leq n-1$ and $\sigma = ad(k)$ where $k = \operatorname{diag}\{k_1, k_2\}$, $k_1 \in U(r)$, $k_2 \in O(2n-2r)$, $\det k_2 = -1$.

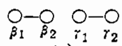
In case (c), $k \in O(2n)$ has square $\pm I$ and determinant -1, so $k^2 = I$. Thus $k_i^2 = I$. Now $G^* = SO^{2s+t}(2n)/Z_2$ where $K^* = \{U^s(r) \times SO^t(2n-2r)\}/Z_2$ with t odd.

In case (b), $\sigma = \nu \cdot ad(g)$ where $g \in G$ and ν is complex conjugation of matrices. $1 = \sigma^2 = ad(\bar{g}) \cdot ad(g) = ad({}^tg^{-1}) \cdot ad(g) = ad({}^tg^{-1} \cdot g)$ shows $g = c\,{}^tg$ for some complex number c with $c^n = 1$. Now ${}^t({}^tg) = g$ shows $c = \pm 1$. g has form $\begin{pmatrix} O & A \\ B & O \end{pmatrix}$ in $r \times r$ blocks because σ interchanges the two $U(r)$ factors of K, so $g = \begin{pmatrix} O & A \\ B & O \end{pmatrix}$ with $B = c\,{}^tA$. Now G^* is $SL(n, R)/Z_2$ if $c = 1$, $SL(r, Q)/Z_2$ if $c = -1$, and K^* is the image of $\{g \in GL(r, C) : |\det g| = 1\}$ in any case.

In case (a), let σ_0 be the automorphism of G defined on a Weyl basis by



. If $\sigma = \sigma_0$ then $G^* = E_{6,F_4}$ and $K^* = \{SL(3, C) \times SU(3)\}/Z_3$.

As K is its own normalizer in G, $\sigma = \sigma_0\nu$ with $\nu = ad(g)$ for some $g \in K$ fixed by σ_0. We note that $K^{\sigma_0} = \{SU(3) \times SU(3)\}/Z_3$ has diagram 

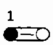
where $\beta_1 = -(\phi_1 + \phi_5) - 2(\phi_2 + \phi_4 + \phi_6) - 3\phi_3$, $\beta_2 = \phi_6$, $\gamma_1 = \frac{1}{2}(\phi_2 + \phi_4)$ and $\gamma_2 = \frac{1}{2}(\phi_1 + \phi_5)$; so the nonzero vertices of its fundamental simplex are $u_1 = -v_3$,

$u_2 = 2v_6 - 2v_3$, $w_1 = 2(v_2 + v_4) - 4v_3$ and $w_2 = v_1 + v_5 - 2v_3$. Now we need only consider the cases $\nu = ad(\exp 2\pi\sqrt{-1}\,x)$ for $x = \frac{1}{2}u_i,\ \frac{1}{2}w_i,\ \frac{1}{2}(u_1 + w_1)$ and $\frac{1}{2}(u_1 + w_2)$. If $x = \frac{1}{2}w_i$ then $K^* = \{SL(3, C) \times SU(3)\}/Z_3$, and then $G^* = E_{6, F_4}$ because $A_2A_2 \not\subset C_4$. If x is $\frac{1}{2}u_i$ or $\frac{1}{2}(u_i + w_j)$, then K^* is $\{SL(3, C) \times SU^1(3)\}/Z_3$ and we run through a list of roots to compute dimensions of intersections of eigenspaces as $\dim\{\mathfrak{E}(\sigma_0, 1) \cap \mathfrak{E}(\nu, 1)\} = 24$ and $\dim\{\mathfrak{E}(\sigma_0, -1) \cap \mathfrak{E}(\nu, -1)\} = 12$; thus $\dim \mathfrak{E}(\sigma, 1) = 36$ and $G^* = E_{6, C_4}$.

This completes our analysis of the case where rank $G =$ rank K. We go on to table 3 of Theorem 6.1.

Let $G = L \times L \times L$ with L simple and $K = \{(g_1, g_2, g_3) \in G : g_1 = g_2 = g_3\}$. If σ preserves each factor of G then $\sigma(K) = K$ says that $\sigma = \nu \times \nu \times \nu$ for some involutive automorphism ν of L; then $G^* = L^* \times L^* \times L^*$ and K is L^* embedded diagonally, where L^* is the form of L defined by the involution ν. If σ permutes factors of G, then we may take $\sigma(g_1, g_2, g_3) = (\nu_2 g_2, \nu_1 g_1, \nu_3 g_3)$, $\nu_i^2 = 1$, from $\sigma^2 = 1$; and further $\nu_1 = \nu_2 = \nu_3$ as $\sigma(K) = K$, so σ acts by $(g_1, g_2, g_3) \to (\nu g_2, \nu g_1, \nu g_3)$; then $G^* = L^C \times L^*$ and $K^* \cong L^*$ is given by $K^* = \{(g, g') \in L^C \times L^* : g = g'\}$. In all cases K^* has linear isotropy representation $ad_{K^*} \oplus ad_{K^*}$, whose commuting algebra is the algebra of 2×2 real matrices; so the invariant almost complex structures on X^* are in 1 to 1 correspondence with the 2×2 real matrices of square $-I$.

For the remaining two cases we replace $Spin(8)$ by $SO(8)$; this is permissible because $\sigma^2 = 1$ says that σ is conjugate in the automorphism group of G to conjugation by some element s in the full orthogonal group $O(8)$.

$SO(8)/G_2$. $\sigma(K) = K$ says that s permutes the irreducible summands of the representation of G_2 on R^8. Thus $R^8 = R^1 \oplus R^7$ under G_2 and $s = (\pm 1) \oplus s'$, $s' \in O(7)$. $\sigma^2 = 1$, so $s^2 = \pm I$, and now $s^2 = I$. $\sigma|_{G_2}$ is necessarily inner, so $\sigma|_{G_2} = ad(t)$ for some $t \in G_2$ of square 1. If $t \neq 1$, so $s' = \begin{pmatrix} -I_r & \\ & I_{7-r} \end{pmatrix}$ with $1 \leq r \leq 6$, then $G_2^* \subset SO^r(7)$ and maximal compact subgroups $SO(4) \subset SO(r) \times SO(7 - r)$. As $SO(4)$ contains a Cartan subgroup of G_2^*, and as the representation  of the latter on R^7 has O as a weight of multiplicity 1, we have $r > 2$ and $7 - r > 2$. Now r is 3 or 4. Changing s to $-s$ if necessary we may assume $r = 3$. Then s is $\begin{pmatrix} -I_4 & \\ & I_4 \end{pmatrix}$ or $\begin{pmatrix} -I_3 & \\ & I_5 \end{pmatrix}$, σ being inner in the first case and outer in the second, provided $\sigma \neq 1$:

s	$\pm I_8$	$\pm\begin{pmatrix} -1 & \\ & I_7 \end{pmatrix}$	$\pm\begin{pmatrix} -I_4 & \\ & I_4 \end{pmatrix}$	$\pm\begin{pmatrix} -I_3 & \\ & I_5 \end{pmatrix}$
G^*	$SO(8)$	$SO^1(8)$	$SO^4(8)$	$SO^3(8)$
K^*	G_2		G_2^*	

In all cases the isotropy representation of K^* has commuting algebra (2×2 real matrices), so the G^*-invariant almost complex structures on X^* are in 1 to 1 correspondence with 2×2 real matrices of square $-I$.

$\boldsymbol{SO}(8)/ad\boldsymbol{SU}(3)$. $\sigma|_K$ is inner because σ cannot interchange the irreducible summands $\circ\!-\!\overset{3}{\circ}$ and $\overset{3}{\circ}\!-\!\circ$ of the linear isotropy representation. Now $\sigma|_K = ad(ad(v))|_K$ for some $v \in \boldsymbol{SU}(3)$. If $v \neq I_3$ then v^2 scalar allows us to assume $v = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$, so $ad(v) = \begin{pmatrix} -I_4 & \\ & I_4 \end{pmatrix} \in \boldsymbol{SO}(8)$. In either case Schur's Lemma and $ad(v)^2 = I_8$ say $s^{-1} \cdot ad(v) = \pm I_8$. Now $s = \pm I_8$, $G^* = \boldsymbol{SO}(8)$, $K^* = ad\boldsymbol{SU}(3)$; or $s = \pm \begin{pmatrix} -I_4 & \\ & I_4 \end{pmatrix}$, $G^* = \boldsymbol{SO}^4(8)$, $K^* = ad\boldsymbol{SU}^1(3)$.

This completes the run through the three tables of Theorem 6.1. There is no redundancy for the cases rank $G >$ rank K. Now we need the following lemma, which eliminates redundancy for the cases of equal rank.

7.15. Lemma. *Let rank $G =$ rank K, G/K listed in Theorem* 6.1. *Let σ_i be involutive automorphisms of G, which preserve K and invariant almost complex structures on G/K. Let G_i^* and K_i^* be the corresponding real forms of G and K. Suppose* (i) *K_1^* and K_2^* are of the same type,* (ii) *G_1^* and G_2^* are of the same type. Then there is an automorphism β of G which preserves K such that $\sigma_2 = \beta\sigma_1\beta^{-1}$.*

Proof. By (ii) there is an automorphism β' of G such that $\sigma_2 = \beta'\sigma_1\beta'^{-1}$. Now we must find β in the form $\alpha\beta'$ where α commutes with σ_2.

Define $\sigma = \sigma_2$ and $\theta_2 = \theta$. Define $\theta_1 = \beta'^{-1}\theta\beta'$ and $K_i = G^{\theta_i}$. Now (i) says that $K_1^\sigma = (G^\sigma)^{\theta_1}$ and $K_2^\sigma = (G^\sigma)^{\theta_2}$ are of the same type, thus conjugate by an inner automorphism $ad(a)$ of G^σ. Let $\alpha = ad(a)$ on G; $a \in G^\sigma$ says $\sigma_2\alpha = \alpha\sigma_2$; replace β' by $\beta = \alpha\beta'$; we still have $\sigma_2 = \beta\sigma_1\beta^{-1}$ but now $(G^\sigma)^{\theta_1} = (G^\sigma)^{\theta_2}$. Thus $\theta_1\theta_2^{\pm 1} = ad(v)$ where v is central in G^σ, $v^3 = 1$. If $v \neq 1$, then G^σ is a hermitian symmetric subgroup of G, so the center of G^σ is a circle group; then $\theta_i = ad(v^{\pm 1})$ and $\theta_1 = \theta_2^{\pm 1}$, i.e. $\beta\theta\beta^{-1} = \theta^{\pm 1}$, i.e. $\beta(K) = K$. q.e.d.

The final step is to check the global form of each of the entries of the table of our theorem. There we must check that G^*/K^* is simply connected and that G^* acts effectively. For the first, $\pi_1(G^*/K^*) = \pi_1(A/B)$ where B is a maximal compact subgroup of K^* contained in a maximal compact subgroup A of G^*. For the second, using the fact that $\mathfrak{G}^*$ has no nonzero ideal contained in $\mathfrak{K}^*$, we need only check $K^* \cap Z^* = \{1\}$ where Z^* is the center of G^*. These small calculations are left to the reader. q.e.d.

Theorem 7.10 extends Theorem 6.1 to the "noncompact case." To extend Theorem 6.4 we need an appropriate version of the connectedness of the isotropy subgroup as mentioned in Proposition 4.1.

7.16. Lemma. *Let $X = G/H$ be an effective coset space, where G is a connected reductive Lie group and H is a closed reductive subgroup of maximal rank. Choose maximal compact subgroup $L \subset K$ of $H \subset G$ and suppose* *rank* $L =$ *rank* K. *If* X *carries a* G*-invariant almost complex structure, then* H *is connected.*

Proof. $\mathfrak{H}$ contains the center of $\mathfrak{G}$ because it has maximal rank, so H contains the identity component of the center of G. Now G is semisimple because it acts effectively on X. Let $\mathfrak{S}$ be the centralizer of $\mathfrak{K}$ in $\mathfrak{G}$. Define $\mathfrak{K}' = \mathfrak{K} + \mathfrak{S}$ and $\mathfrak{L}' = \mathfrak{L} + \mathfrak{S}$. $\mathfrak{K}'$ is a compactly embedded subalgebra of $\mathfrak{G}$ so $\mathfrak{L}'$ is a compactly embedded subalgebra of $\mathfrak{H}$. The linear isotropy representation of H_0 is faithful because G is effective on X; thus the analytic subgroup of H with Lie algebra L' is compact; it follows that $\mathfrak{L}' = \mathfrak{L}$. Now $\mathfrak{S} \subset \mathfrak{L}$, so $\mathfrak{S} \subset \mathfrak{K}$, and this shows that $\mathfrak{L}$ and $\mathfrak{K}$ are maximal compactly embedded subalgebras of $\mathfrak{H}$ and $\mathfrak{G}$. In particular K contains the center of G, L contains the center of H, and L_0 contains the center of H_0.

Let $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ be the orthogonal decomposition under the Killing form. Decompose $\mathfrak{M}^C = \mathfrak{M}^+ + \mathfrak{M}^-$ into $\pm(\sqrt{-1})$-eigenspaces of the invariant almost complex structure. Now let Z denote the center of H_0, so H_0 and L_0 are the respective identity components of the centralizer of Z in G and K. Choose a Cartan involution σ of G which preserves H_0. Then $\sigma(Z) = Z$ and K is the fixed point set of σ on G. Now $\mathfrak{K} = \mathfrak{L} + \mathfrak{N}$ where $\mathfrak{N} = \mathfrak{K} \cap \mathfrak{M}$. Define $\mathfrak{N}^+ = \mathfrak{K}^C \cap \mathfrak{M}^+ = \mathfrak{N}^C \cap \mathfrak{M}^+$ and $\mathfrak{N}^- = \mathfrak{K}^C \cap \mathfrak{M}^- = \mathfrak{N}^C \cap \mathfrak{M}^-$, so $\mathfrak{N}^C = \mathfrak{N}^+ + \mathfrak{N}^-$ defines a K-invariant almost complex structure on K/L. Proposition 4.1 says that L is connected. As L meets every component of H, now H is connected. q.e.d.

Now we can complete Theorem 7.10 to a structure-classification theorem which extends Theorem 6.4 to the noncompact case.

7.17. Theorem. *The coset spaces* $X = G/H$ *with the properties* (i) G *is a connected reductive Lie group acting effectively,* (ii) $\mathfrak{H} = \mathfrak{G}^\theta$ *where* θ *is an automorphism of order* 3 *on* $\mathfrak{G}$, *and* (iii) X *carries a* G*-invariant almost complex structure, are precisely the spaces* $(X_0 \times X_1 \times \cdots \times X_r)/\Gamma = [(G_0 \times G_1 \times \cdots \times G_r)/\Gamma]/H$ *constructed as follows.*

X_0 *is a complex euclidean space,* G_0 *is its translation group, and* $H_0 = \{1\} \subset G_0$;

$r \geq 0$ *is an integer. If* $1 \leq i \leq r$, *then* $X_i = G_i/H_i$ *is one of the spaces listed in Theorem* 7.10, *and* Z_i *denotes the center of* G_i;

Γ *is arbitrary discrete subgroup of* $G_0 \times Z_1 \times \cdots \times Z_r$;

$G = (G_0 \times G_1 \times \cdots \times G_r)\Gamma$ *and* H *is the image of* $H_0 \times H_1 \times \cdots \times H_r$ *in* G.

Remark. Z_i is trivial if rank $G_i =$ rank H_i, i.e. if $X_i = G_i/H_i$ is listed in Table 7.11, 7.12 or 7.13. If rank $G_i >$ rank H_i, i.e. if $X_i = G_i/H_i$ is listed in Table 7.14, then Z_i is:

$\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ if G_i is $\boldsymbol{Spin}(8)$, $\boldsymbol{Spin}^1(8)$, $\boldsymbol{Spin}^3(8)$, $\boldsymbol{Spin}^4(8)$ or $\boldsymbol{Spin}(8, C)$;

$\boldsymbol{Z}_2$ if G_i is $\boldsymbol{SO}^4(8)$;

$Z^* \times Z^*$ if $G_i = [L^* \times L^* \times L^*]/\delta Z^*$, Z^* if $G_i = [L^C \times L^*]/\bar{\delta} Z^*$,
$Z \times Z$ if $G_i = [L^C \times L^C \times L^C]/\delta Z$.

Proof. The proof is identical to the proof of Theorem 6.4, except that Theorem 7.10 substitutes for Theorem 6.1 and Lemma 7.16 for Proposition 4.1.

7.18. Corollary. *The coset spaces $X = G/H$ with the properties* (i) *G is a connected reductive Lie group acting effectively,* (ii) $\mathfrak{H} = \mathfrak{G}^\theta$ *where θ is an automorphism of order* 3 *on* $\mathfrak{G}$, (iii) *X carries a G-invariant almost complex structure, and* (iv) *X is locally a product of coset spaces with $\boldsymbol{R}$-irreducible linear isotropy subgroup, are precisely the spaces $(X_0 \times X_1 \times \cdots \times X_r)/\Gamma$ which are listed in Theorem* 7.17 *and satisfy the additional condition:*

If $1 \leq i \leq r$ then X_i is listed in the tables of Theorem 7.10 *with $N = 2$.*

Proof. X is listed in Theorem 7.17 and this is equivalent to conditions (i), (ii) and (iii). Condition (iv) says, precisely, that $1 \leq i \leq r$ implies, in the notation of Theorem 7.17, that the linear isotropy representation β_i of H_i is $\boldsymbol{R}$-irreducible. If β_i is $\boldsymbol{R}$-irreducible then of course $N = 2$. If $N = 2$ then β_i cannot decompose into summands stable under an almost complex structure, so $\beta_i = \pi_i \oplus \bar{\pi}_i$ with π_i absolutely irreducible; then β_i is $\boldsymbol{R}$-irreducible, for reality of π_i would imply (cf. Table 7.14) $N = \infty$. q.e.d.

We have been implicitly using the fact that Theorem 4.3 extends without change to the case where K is a connected reductive subgroup of maximal rank in a connected reductive Lie group G. At this point we should note, for purposes of § 8, that Theorem 4.7 extends without change of the case where K is the identity component of the centralizer of a connected subgroup of a Cartan subgroup of a connected reductive group G, and that Theorem 4.5 and Corollary 4.6 extend to the reductive case with the restrictions that T remains compact and we use restricted Weyl groups.

8. Types of homogeneous almost Hermitian manifolds

In this section and the next we give a detailed description of the almost hermitian geometry of the almost complex manifolds of §§ 4 through 7. The general results are given here in § 8; § 9 is concerned with somewhat more delicate results involving calculations with the root systems of the relevant Lie algebras.

We first describe several conditions for almost hermitian manifolds which are weaker than the kaehler condition. We then prove a series of theorems relating those conditions, for a homogeneous almost hermitian metric on a reductive coset space G/K, to criteria concerning whether $\mathfrak{K}$ is the fixed point set of an automorphism of order 3 of $\mathfrak{G}$.

Let M be a C^∞ real differentiable manifold and $\mathscr{X}(M)$ the Lie algebra of vector fields on M. We assume that M possesses an almost complex structure

J and a pseudo-riemannian metric tensor field $(,)$ which satisfy $(JX, JY) = (X, Y)$ for all $X, Y \in \mathscr{X}(M)$. The *kaehler form* of J and $(,)$ is the 2-form F defined by $F(X, Y) = (JX, Y)$ for all $X, Y \in \mathscr{X}(M)$. Let $ds^2 = (,) + \sqrt{-1}\, F$. Then the existence of ds^2 on M is equivalent to the existence of compatible $(,)$ and J. We say that (M, ds^2) is an *almost hermitian manifold* and that ds^2 is an *almost hermitian metric* on M.

Assume that (M, ds^2) is almost hermitian and let ∇ denote the riemmanian connection of the pseudo-riemannian metric $(,)$ determined by ds^2. If J is the almost complex structure determined by ds^2, we say that (M, ds^2) is *kaehlerian* if $\nabla_X(J) = 0$ for all $X \in \mathscr{X}(M)$, *almost kaehlerian* if $dF = 0$, *nearly kaehlerian* if $\nabla_X(J)(X) = 0$ for all $X \in \mathscr{X}(M)$, *quasi-kaehlerian* if

$$\nabla_X(J)(Y) + \nabla_{JX}(J)(JY) = 0$$

for all $X, Y \in \mathscr{X}(M)$, *semi-kaehlerian* if $\delta F = 0$, and *hermitian* if J is integrable, i.e. if M is a complex manifold relative to J. Let $\mathscr{K}$, $\mathscr{A}\mathscr{K}$, $\mathscr{N}\mathscr{K}$, $\mathscr{Q}\mathscr{K}$, $\mathscr{S}\mathscr{K}$ and $\mathscr{H}$ denote the classes of kaehler, almost kaehler, nearly kaehler, quasi-kaehler, semi-kaehler, and hermitian manifolds, respectively. In [5] it is shown that the following inclusions hold between the various classes:

$$\mathscr{K} \begin{matrix} < \mathscr{A}\mathscr{K} < \\ < \mathscr{N}\mathscr{K} < \end{matrix} \mathscr{A}\mathscr{K} \cup \mathscr{N}\mathscr{K} < \mathscr{Q}\mathscr{K} < \mathscr{Q}\mathscr{K} \cup (\mathscr{S}\mathscr{K} \cap \mathscr{H}) < \mathscr{S}\mathscr{K} ,$$

$$\mathscr{K} < \mathscr{S}\mathscr{K} \cap \mathscr{H} \begin{matrix} < \mathscr{S}\mathscr{K} < \\ < \mathscr{H} < \end{matrix} \mathscr{S}\mathscr{K} \cup \mathscr{H} < \mathscr{A}\mathscr{H} .$$

Here $<$ denotes strict inclusion and $\mathscr{A}\mathscr{H}$ stands for the class of all almost hermitian manifolds. Furthermore, $\mathscr{K} = \mathscr{H} \cap \mathscr{Q}\mathscr{K} = \mathscr{A}\mathscr{K} \cap \mathscr{N}\mathscr{K}$ so that all possible inclusions are determined.

Now we consider the above conditions on a homogeneous almost hermitian manifold (M, ds^2). We assume that $M = G/K$ is a reductive homogeneous space, and that the metric $(,)$ and almost complex structure of (M, ds^2) are both G-invariant. In refering to classifications we will assume that G is connected and acts effectively on M, but in general we make no additional hypotheses on the Lie groups G and K. In particular, we do not assume the pseudo-riemannian metric $(,)$ to be definite. If the isotropy representation of K has no irreducible summand of multiplicity greater than 1, then homogeneity automatically implies the compatibility condition $(JX, JX) = (X, Y)$ for $X, Y \in \mathscr{X}(M)$. Furthermore, we have the following formulas for $X, Y \in \mathscr{X}(M)$:

$$([X, K], Y) = (X, [K, Y]) , \qquad [K, JX] = J[K, X] .$$

If V is a real vector space and $P : V \to V$ is a linear transformation without real eigenvalues, then P determines an almost complex structure J on V in a canonical fashion. Let $\operatorname{Im}\lambda > 0$ and V_λ be the subspace of V^C on which $P - \lambda I$ is nilpotent. Then J is given on $V \cap (V_\lambda + V_{\bar\lambda})$ by the reguirement that $P - \{(\operatorname{Re}\lambda)I + (\operatorname{Im}\lambda)J\}$ be nilpotent. J extends to $V = \sum_{\operatorname{Im}\lambda>0} V \cap (V_\lambda + V_{\bar\lambda})$ by linearity. The linear transformation J of V has square $-I$ and is called the *canonical almost complex structure* determined by P.

Thus an automorphism θ of G of order n for which -1 is not an eigenvalue determines an invariant almost complex structure on G/K if K is the fixed point set of θ. For $n = 3$ or 4 we shall characterize the canonical almost complex structures of the almost hermitian manifolds so obtained.

8.1. Theorem. *Let $M = G/K$ be a (reductive) homogeneous space for which K is the fixed point set of an automorphism θ of order 4, and assume that -1 is not an eigenvalue of the induced action of θ on $\mathfrak{G}$. Then the canonical almost complex structure J determined by θ, together with any compatible metric $(\ ,\ )$, makes G/K into a hermitian symmetric space. Conversely, if G/K is hermitian symmetric, then $\mathfrak{K}$ is the fixed point set of an automorphism of $\mathfrak{G}$ of order n for any $n > 1$.*

Proof. For the necessity let P be the induced action of θ on $\mathfrak{G}$. On $\mathfrak{M}$ we have $P = J$ so $J[X, Y]_{\mathfrak{M}} = [JX, JY]_{\mathfrak{M}}$ for $X, Y \in \mathfrak{M}$. (Here the subscript denotes the component in $\mathfrak{M}$.) Hence

$$[X, Y]_{\mathfrak{M}} = [J^2X, J^2Y]_{\mathfrak{M}} = J[JX, JY]_{\mathfrak{M}} = -[X, Y]_{\mathfrak{M}} ,$$

and so $[\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{K}$. Thus G/K is hermitian symmetric.

Conversely, if G/K is irreducible hermitian symmetric, then K has a 1-dimensional center Z. It is not hard to see that any element in Z of order n ($n > 1$) has fixed point set $\mathfrak{K}$. q.e.d.

The characterization of an almost complex structure determined by an automorphism of order 3 is more complicated.

8.2. Theorem. *Let $M = G/K$ be a (reductive) homogeneous space for which $\mathfrak{K}$ is the fixed point set of an automorphism θ of $\mathfrak{G}$ of order 3. Then the canonical invariant almost complex structure J determined by θ satisfies*

$$[JX, Y]_{\mathfrak{M}} = -J[X, Y]_{\mathfrak{M}} , \tag{8.3}$$

$$[X, Y]_{\mathfrak{K}} = [JX, JY]_{\mathfrak{K}} \tag{8.4}$$

for all $X, Y \in \mathfrak{M}$. Conversely, if $M = G/K$ has an invariant almost complex structure satisfying (8.3) *and* (8.4), *then $\mathfrak{K}$ is the fixed point set of an automorphism of $\mathfrak{G}$ of order* 3.

Proof. For the necessity let P denote the induced action of θ on $\mathfrak{G}$. The canonical almost complex structure on $\mathfrak{M}$ determined by P is given by

$$P_{\mathfrak{M}} = -\frac{1}{2} I + \frac{\sqrt{3}}{2} J . \tag{8.5}$$

Since $P[X, K] = [PX, K]$ for $X \in \mathfrak{M}$ and $K \in \mathfrak{K}$, it follows that J is invariant. Furthermore,

$$(8.6)\qquad \begin{aligned} P[X, Y] - [PX, PY] = \frac{3}{4}[X, Y]_{\mathfrak{K}} - \frac{3}{4}[X, Y]_{\mathfrak{M}} + \frac{\sqrt{3}}{2}J[X, Y]_{\mathfrak{M}} \\ - \frac{3}{4}[JX, JY] + \frac{\sqrt{3}}{4}[JX, Y] + \frac{\sqrt{3}}{4}[X, JY] . \end{aligned}$$

In particular,

$$(8.7)\qquad P[X, JX] - [PX, PJX] = -\frac{3}{2}[X, JX]_{\mathfrak{M}} + \frac{\sqrt{3}}{2}J[X, JX]_{\mathfrak{M}} .$$

Since P is an automorphism of $\mathfrak{G}$ the left hand sides of (8.6) and (8.7) vanish. From (8.7) it follows that $[X, JX]_{\mathfrak{M}} = 0$. Hence (8.6) reduces to

$$(8.8)\qquad \begin{aligned} 0 = \frac{3}{4}([X, Y]_{\mathfrak{K}} - [JX, JY]_{\mathfrak{K}}) + \frac{\sqrt{3}}{4}([JX, Y]_{\mathfrak{K}} + [X, JY]_{\mathfrak{K}}) \\ + \frac{\sqrt{3}}{2}(J[X, Y]_{\mathfrak{M}} + [JX, Y]_{\mathfrak{M}}) . \end{aligned}$$

Thus we get (8.3). Furthermore, (8.4) is obtained by substituting JY for Y in (8.8) and subtracting the result from (8.8).

Conversely, suppose (8.3) and (8.4) hold. Define $P: \mathfrak{G} \to \mathfrak{G}$ by (8.5) and the requirement that P be the identity on $\mathfrak{K}$. From (8.3) we have $[JX, Y]_{\mathfrak{M}} = [X, JY]_{\mathfrak{M}}$ for $X, Y \in \mathfrak{M}$, and (8.3) and (8.4) imply (8.8). Thus (8.6) becomes $P[X, Y] = [PX, PY]$ for $X, Y \in \mathfrak{M}$. Furthermore, since J is invariant, $P[X, K] = [PX, K]$ for $X \in \mathfrak{M}$, $Y \in \mathfrak{K}$. Therefore P is an automorphism of $\mathfrak{G}$ with fixed point set $\mathfrak{K}$. {Consequently, if G is simply connected, P determines an automorphism of G of order 3 whose fixed point set is K_0.} q.e.d.

The next theorem shows that it is sometimes possible to determine the class of a homogeneous almost hermitian manifold (M, ds^2) even if the metric $(\,,\,)$ is not assumed to be obtained by restriction of M and translation over G/K of a bi-invariant bilinear form on $\mathfrak{G}$.

8.9. Theorem. *Let (M, ds^2) be a reductive homogeneous almost hermitian manifold, $M = G/K$.*

(i) *If J satisfies (8.3) then $(M, ds^2) \in \mathcal{QK}$.*

(ii) *If the isotropy representation of K has no invariant 1-dimensional subspaces, then $(M, ds^2) \in \mathcal{SK}$. This holds, for example, if the isotropy representation is irreducible or if G and K are reductive Lie groups of equal rank.*

Proof. For (i) we note that the riemannian connection ∇ of M is given by the formula

$$(8.10)\qquad 2(\nabla_X Y, Z) = -(X, [Y, Z]) - (Y, [X, Z]) + (Z, [X, Y])$$

for $X, Y, Z \in \mathfrak{M}$. Because of (8.3) we have

$$\nabla_X(F)(Y, Z) = +(X, J[Y, Z]) + (Y, J[X, Z]) - (Z, J[X, Y]) .$$

Again on account of (8.3) it follows that

$$\nabla_X(F)(Y, Z) + \nabla_{JX}(F)(JY, Z) = 0.$$

Finally (ii) follows from the fact that δF is invariant under the isotropy representation of K. q.e.d.

Next we assume that (M, ds^2) has a metric $(\, , \,)$ which is the projection of a bi-invariant metric on G. This holds, for example, if the isotropy representation of K is irreducible. We have then

$$([X, Y], Z) = (X, [Y, Z]) , \qquad ([X, Y], K) = (X, [Y, K])$$

for $X, Y, Z \in \mathfrak{M}$ and $K \in \mathfrak{K}$. Furthermore, the riemannian connection of M is given by $\nabla_X Y = \frac{1}{2}[X, Y]_{\mathfrak{M}}$ for $X, Y \in \mathfrak{M}$.

8.11. Theorem. *Let (M, ds^2) be a reductive homogeneous almost hermitian manifolds with $M = G/K$ such that the metric $(\, , \,)$ of M is a projection of a bi-invariant metric on G. Then the following conditions are equivalent:*

(i) $(M, ds^2) \in \mathcal{NK}$.

(ii) $[X, JX] \in \mathfrak{K}$ *for all* $X, Y \in \mathfrak{M}$.

(iii) $\mathfrak{K}$ *is the fixed point set of an automorphism of* $\mathfrak{G}$ *of order* 3.

Proof. We have $\nabla_X(J)(X) = \frac{1}{2}[X, JX]_{\mathfrak{M}}$, and so (i) and (ii) are equivalent. Furthermore, (ii) is equivalent to equation (8.3). Bi-invariance of $(\, , \,)$ implies that (8.4) holds. The rest of the implications follow from these facts.

8.12. Theorem. *Let the metric of the homogeneous almost hermitian manifold (M, ds^2), $M = G/K$ reductive, be the projection of a bi-invariant metric on G. Then*

(i) $(M, ds^2) \in \mathcal{NK}$ *if and only if* $(M, ds^2) \in \mathcal{QK}$,

(ii) $(M, ds^2) \in \mathcal{K}$ *if and only if* $(M, ds^2) \in \mathcal{AK}$.

Proof. We have

$$\nabla_X(J)(X) + \nabla_{JX}(J)(JX) = [X, JX]_{\mathfrak{M}} = 2\nabla_X(J)(X) .$$

If $(M, ds^2) \in \mathcal{QK}$, the left hand side of the first equation is zero and so $(M, ds^2) \in \mathcal{NK}$. Since we always have $\mathcal{NK} \subset \mathcal{QK}$, (i) follows. Furthermore, $\mathcal{AK} \subset \mathcal{QK}$ and $\mathcal{AK} \cap \mathcal{NK} = \mathcal{K}$; hence (ii) follows. q.e.d.

The possible classes for a homogeneous almost hermitian manifold (M, ds^2), whose almost complex structure J is canonically determined by an automorphism of order 3, are summarized by the following theorem.

8.13. Theorem. *Suppose $M = G/K$ is a (reductive) homogeneous space* *and $\mathfrak{K}$ is the fixed point set of an automorphism θ of $\mathfrak{G}$ of order* 3. *Let ds^2 be a G-invariant almost hermitian metric on M whose associated almost complex structure is the canonical one determined by θ. Then*

(i) $(M, ds^2) \in \mathscr{Q}\mathscr{K}$;

(ii) *if the metric* (,) *of M is induced from a bi-invariant metric on G, then* $(M, ds^2) \in \mathscr{N}\mathscr{K}$;

(iii) *under the hypothesis of* (ii), *the following are equivalent*: (a) $(M, ds^2) \in \mathscr{H}$; (b) $(M, ds^2) \in \mathscr{K}$; (c) *M is hermitian symmetric with respect to J.*

Proof. (i) follows from (8.9), and (ii) is a consequence of (8.11). For (iii) we note that (c) implies (a) and (b), and (a) and (b) are equivalent by (i). Furthermore, if $(M, ds^2) \in \mathscr{K}$ and the metric of M is induced from a bi-invariant metric of G, we have $J[X, Y]_{\mathfrak{M}} = [JX, Y]_{\mathfrak{M}}$ for $X, Y \in \mathfrak{M}$. That M is hermitian symmetric now follows from (8.3). q.e.d.

A weak version of Theorem 8.13 can be proved in the general case (where the almost complex structure is not assumed to be the canonical one):

8.14. Theorem. *Let (M, ds^2) be a homogeneous almost hermitian manifold such that $M = G/K$, where G is a reductive Lie group and $\mathfrak{K} = \mathfrak{G}^\theta$ for some automorphism θ of order* 3 *on $\mathfrak{G}$. Then $(M, ds^2) \in \mathscr{S}\mathscr{K}$.*

If M is compact and θ induces an outer automorphism on the semisimple part of $\mathfrak{G}$, then $(M, ds^2) \notin \mathscr{H}$, $(M, ds^2) \notin \mathscr{K}$ and $(M, ds^2) \notin \mathscr{A}\mathscr{K}$.

Proof. Without loss of generality we may assume G connected and effective on M. Then M is one of the spaces $\overline{M}/\Gamma$, $\overline{M} = M_0 \times M_1 \times \cdots \times M_r$, of Theorem 7.17. We are examining conditions determined by integrability of the almost complex structure J of ds^2 and by the differential dF and the codifferential δF of the kaehler form F of ds^2. Those properties are local so we may replace M by $\overline{M}$. After having done this we have $M = M_0 \times M_1 \times \cdots \times M_r$, $G = G_0 \times G_1 \times \cdots \times G_r$, $K = K_0 \times K_1 \times \cdots \times K_r$ and $M_i = G_i/K_i$, where M_0 is a complex euclidean space and the other M_i are listed in the tables of Theorem 7.10. ds^2 is the direct sum of G_i-invariant almost hermitian metrics ds_i^2 on the M_i; if J_i and F_i denote the almost complex structure and kaehler form of ds_i^2, then J is the direct sum of the J_i and F is the direct sum of the F_i.

δF_i is a G_i-invariant 1-form on M_i. Let $\mathfrak{M}_i$ denote the complement to $\mathfrak{K}_i$ in $\mathfrak{G}_i$. If $\delta F_i \neq 0$, then $ad_{G_i}|_{K_i}$ must have a trivial subrepresentation on $\mathfrak{M}_i$. If rank $K_i =$ rank G_i, Theorem 4.3 show that there is no such trivial subrepresentation. If rank $K_i <$ rank G_i but M_i is not a complex euclidean space, the same fact follows from Theorem 5.10. Now $\delta F_i = 0$ for $i > 0$. But ds_0^2 is stable under a nonhomogeneous indefinite unitary group on the complex euclidean space M_0, and the isotropy subgroup, which is an indefinite unitary group, is irreducible; as before it follows that $\delta F_0 = 0$. Now $\delta F = \delta(F_0 \oplus F_1 \oplus \cdots \oplus F_r) = \delta F_0 \oplus \cdots \oplus \delta F_r = 0$. That proves $(M, ds^2) \in \mathscr{S}\mathscr{K}$.

Now suppose that M was compact before we replaced it with $\bar{M}$, and the θ induces an outer automorphism of the semisimple part of G. Then G_i is a compact for $i > 0$ and we may assume $\theta|_{G_1}$ to be an outer automorphism. Note, from Theorem 6.1 or Table. 7.14, that K_1 is semisimple. K_1 cannot be the semisimple part of the centralizer of a toral subgroup of G_1 because $ad_{G_1}|_{K_1}$ has no trivial subrepresentation on $\mathfrak{M}_1$; it follows [10] that J_1 is not integrable; then J is not integrable, so $(M, ds^2) \notin \mathscr{H}$ and $(M, ds^2) \notin \mathscr{K}$.

Retain the assumptions of the preceding paragraph. Suppose $dF_1 = 0$. If $F_1 = d\beta$ for some 1-form β on $M_1 = G_1/K_1$, let β' denote the Haar integral average of β over G_1. Then β' is a G_1-invariant 1-form on M_1 and

$$\begin{aligned} d\beta' &= d\int_{G_1} g^*\beta d\mu(g) = \int_{G_1} d(g^*\beta)d\mu(g) = \int_{G_1} g^*d\beta d\mu(g) \\ &= \int_{G_1} g^*F d\mu(g) = \int_{G_1} F d\mu(g) = F \,. \end{aligned}$$

Thus we may assume β to be G_1-invariant. But then $\beta = 0$ because $ad_{G_1}|_{K_1}$ has no trivial subrepresentation on $\mathfrak{M}_1$. That contradiction shows $F_1 \neq d\beta$ for any 1-form β. In other words, the closed 2-form F_1 represents a nonzero cohomology class in $H^2(M_1; \boldsymbol{R})$. Duality and the Hurewicz Theorem then show that the homotopy group $\pi_2(M_1)$ is infinite. But $\pi_1(K_1)$ is finite because K_1 is semisimple, and we have the exact sequence $0 = \pi_2(G_1) \to \pi_2(M_1) \to \pi_1(K_1)$. This contradiction shows $dF_1 \neq 0$. Now $dF \neq 0$, so $(M, ds^2) \notin \mathscr{K}$ and $(M, ds^2) \notin \mathscr{AK}$. q.e.d.

Let $M = G/K$ be a reductive homogeneous space with $\boldsymbol{R}$-irreducible linear isotropy representation. Then every invariant riemannian metric is induced from a bi-invariant metric on G, under the mild condition that G is the translation group if M is an euclidean space or a circle. Let ds^2 be an invariant almost hermitian metric on M. Then the almost complex structure J is unique up to sign [12], and Theorem 8.11 shows that $(M, ds^2) \in \mathscr{NK}$ if and only if $\mathfrak{K}$ is the fixed point set $\mathfrak{G}^\theta$ for some automorphism θ or order 3 on $\mathfrak{G}$. This situation persists under products and under quotient by discrete central subgroup of G. In summary, we have

8.15. Theorem. *If $M = G/K$ is one of the spaces of Corollary 7.18, and ds^2 is any G-invariant almost hermitian metric on M, then $(M, ds^2) \in \mathscr{NK}$.*

By way of contrast we have

8.16. Theorem. *Let $M = G/K$ be a reductive coset space with an invariant almost hemitian metric ds^2. Suppose that $\mathfrak{K}$ is not the fixed point set of an automorphism of order 3 of $\mathfrak{G}$, and that the linear isotropy repeesentation of $\mathfrak{K}$ is irreducible. Then $(M, ds^2) \in \mathscr{SK}$ and $(M, ds^2) \notin \mathscr{QK}$.*

Proof. (M, ds^2) is semikaehlerian because the linear isotropy representation of K cannot have δF as a nonzero invariant. If (M, ds^2) were quasikaehlerian it would be nearly kaehlerian by Theorem 8.12, and then Theorem

8.11 would force $\mathfrak{K}$ to be the fixed point set of an automorphism of order 3 on $\mathfrak{G}$. q.e.d.

The spaces $M = G/K$ which satisfy the hypotheses of Theorem 8.16 have been classified [12, § 13]. The ones for which G is not a complex Lie group are given by $G = \bar{G}/Z$ and $K = \bar{K}Z/Z$, where Z is an arbitrary subgroup of the center $\bar{Z}$ of $\bar{G}$, and all possibilities are given by:

$\bar{G}$	$\bar{Z}$	$\bar{K}$	conditions
$Spin(n^2-1)$	$Z_2 \times Z_2$	$SU(n)/Z_n$	n odd, $n>3$
$SO^{2r(n-r)}(n^2-1)$	Z_2	$SU^r(n)/Z_n$	n odd, $n>3$, $0<2r<n$
$SO^{2r(n-r)}(n^2-1)$	$\{1\}$	$SU(n)/Z_n$	n even, $n>3$, $0\leq 2r \leq n$
E_6	Z_3	$SU(3)/Z_3$	—
simply connected group of type E_{6,A_5A_1}	Z_6	$SU^1(3)/Z_3$	—

{Note that $n = 3$ is excluded from the first two entries of the table by the condition that $\mathfrak{K}$ is not the fixed point set of an automorphism of order 3 of $\mathfrak{G}$.}

The spaces $M = G/K$ satisfying the hypotheses of Theorem 8.16, for which G is a complex Lie group, are the spaces given by $G = A^C/Z$ and $K = B^C Z/Z$, where Z is an arbitrary central subgroup of A (i.e. an arbitrary central subgroup of the complexification A^C), and A/B is either a compact simply connected nonhermitian symmetric coset space or one of the coset spaces listed in [12, Theorem 11.1] for which the linear isotropy representation χ is absolutely irreducible.

9. Invariant almost Hermitian structures on compact homogeneous spaces of positive characteristic

We conclude by studying the types of positive definite invariant almost hermitian metrics ds^2 on homogenous spaces $M = G/K$, where G is a compact connected Lie group and K is a subgroup of maximal rank. Note that $(M, ds^2) \in \mathscr{S}\mathscr{K}$ by Theorem 8.9, and that Theorem 4.5 gives the criterion for whether $(M, ds^2) \in \mathscr{H}$. We find root system criteria for (M, ds^2) to be in the classes $\mathscr{K}$, $\mathscr{A}\mathscr{K}$, $\mathscr{Q}\mathscr{K}$ and $\mathscr{N}\mathscr{K}$, respectively, and we specialize those criteria with the aid of the various classifications of § 4.

Choose a maximal torus T of G which is contained in K. Let Δ_K and Δ denote the respective systems of $\mathfrak{T}^C$-roots of $\mathfrak{K}^C$ and $\mathfrak{G}^C$, and let $\langle\ ,\ \rangle$ denote the Killing form on $\mathfrak{G}^C$. For $\lambda \in \Delta$ we denote by h_λ the element of $\sqrt{-1}\,\mathfrak{T}$ such that $\langle h_\lambda, h\rangle = \lambda(h)$ for all $h \in \mathfrak{T}^C$. Since G acts effectively on M, G is semisimple, and so h_λ is well-defined. Next we choose root vectors $e_\lambda \in \mathfrak{G}_\lambda$ for

$\lambda \in \Delta$ with the following properties: $[h_\lambda, e_\nu] = \langle\lambda, \nu\rangle e_\nu$; $[e_\lambda, e_{-\lambda}] = h_\lambda$; $[e_\lambda, e_\nu] = n_{\lambda,\nu} e_{\lambda+\nu}$ if $\lambda + \nu \in \Delta$; $[e_\lambda, e_\nu] = 0$ if $\lambda + \nu \notin \Delta$. If $\lambda + \nu$ is not a root we define $n_{\lambda,\nu} = e_{\lambda+\nu} = 0$; then we still have $[e_\lambda, e_\nu] = n_{\lambda,\nu} e_{\lambda+\nu}$. The e_λ can be chosen so that $n_{\lambda,\nu}$ is real, and $n_{\lambda,\nu} = -n_{-\lambda,-\nu}$ (see [7, Chapter 3]). Then

$$n_{\lambda,\nu} = -n_{\nu,\lambda} = n_{-\lambda,\lambda+\nu}, \tag{9.1}$$

because of the anticommutativity of $\mathfrak{G}$ and the Jacobi identity.

For all $\lambda \in \Delta$ we set $x_\lambda = e_\lambda - e_{-\lambda} = -x_{-\lambda}$ and $y_\lambda = \sqrt{-1}(e_\lambda + e_{-\lambda}) = y_{-\lambda}$. If Ψ is any system of simple roots and Δ^+ is the corresponding system of positive roots, then

$$\{\sqrt{-1}\, h_\phi;\ x_\lambda;\ y_\lambda : \phi \in \Psi, \lambda \in \Delta^+\}$$

is a basis of $\mathfrak{G}$. In order to compute in this basis we need the following preliminary calculation, which is left to the reader.

9.2. Lemma. *If $\lambda, \nu \in \Delta$ then*

$$[x_\lambda, y_\lambda] = 2\sqrt{-1}\, h_\lambda, \quad [\sqrt{-1}\, h_\lambda, x_\nu] = \langle\lambda, \nu\rangle y_\nu, \quad [\sqrt{-1}\, h_\lambda, y_\nu] = -\langle\lambda, \nu\rangle x_\nu.$$

If, further, $\lambda \neq \pm\nu$, and we make the convention that $e_\alpha = x_\alpha = y_\alpha = 0$ for $\alpha \notin \Delta$, then

$$\begin{aligned}
[x_\lambda, x_\nu] &= n_{\lambda,\nu} x_{\lambda+\nu} - n_{\lambda,-\nu} x_{\lambda-\nu}, \\
[y_\lambda, y_\nu] &= -n_{\lambda,\nu} x_{\lambda+\nu} - n_{\lambda,-\nu} x_{\lambda-\nu}, \\
[x_\lambda, y_\nu] &= n_{\lambda,\nu} y_{\lambda+\nu} + n_{\lambda,-\nu} y_{\lambda-\nu}.
\end{aligned}$$

We can now describe invariant almost hermitian metrics. For this purpose decompose $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$, $\mathfrak{M}^C = \sum_{\lambda \in \Delta - \Delta_K} \mathfrak{G}_\lambda$, and break $\Delta - \Delta_K$ into a disjoint union of subsets Γ_i such that the irreducible representation spaces of K on $\mathfrak{M}^C$ are the spaces $\mathfrak{M}_i = \sum_{\lambda \in \Gamma_i} \mathfrak{G}_\lambda$. We arrange the Γ_i into a sequence $\{\Gamma_1, \Gamma_{-1}; \cdots; \Gamma_r, \Gamma_{-r}; \Gamma_{r+1}; \cdots; \Gamma_{r+s}\}$ where $\Gamma_i = -\Gamma_{-i}$ for $1 \leq i \leq r$ and $\Gamma_i = -\Gamma_i$ for $r + 1 \leq i \leq r + s$. Then the $\boldsymbol{R}$-irreducible representation spaces of K on $\mathfrak{M}$ are the spaces $\mathfrak{N}_i$; $1 \leq i \leq r + s$ given by

$$\begin{aligned}
&\mathfrak{N}_i = (\mathfrak{M}_i + \mathfrak{M}_{-i}) \cap \mathfrak{G}, \quad \mathfrak{N}_i^C = \mathfrak{M}_i + \mathfrak{M}_{-i}, \quad \text{for } 1 \leq i \leq r, \\
&\mathfrak{N}_i = \mathfrak{M}_i \cap \mathfrak{G}, \quad \mathfrak{N}_i^C = \mathfrak{M}_i, \quad \text{for } r + 1 \leq i \leq r + s.
\end{aligned}$$

Then $\mathfrak{N}_i$ has basis $\{x_\lambda, y_\lambda : \lambda \in \Gamma_i \cap \Delta^+\}$, and the Killing form is nondegenerate on each $\mathfrak{N}_i$.

9.3. Proposition. *Let K be a subgroup of maximal rank in a compact connected Lie group G, and retain the notation above.*

1. *The G-invariant pseudo-riemannian metrics on $M = G/K$, viewed as*

$ad_G(K)$-invariant bilinear forms on $\mathfrak{M}$, are just the symmetric bilinear forms $(\ ,\ )$ *with the following properties:* (1a) $\{x_\lambda, y_\lambda : \lambda \in \Delta^+ - \Delta_K\}$ *is a basis of* $\mathfrak{M}$ *consisting of mutually orthogonal vectors (so in particular* $\mathfrak{M} = \sum_{i=1}^{r+s} \mathfrak{N}_i$ *is an orthogonal direct sum), and* (1b) *for each i there is a nonzero real number c_i which defines the bilinear form* $(\ ,\ )$ *on* $\mathfrak{N}_i$ *by the condition* $\|x_\lambda\|^2 = c_i = \|y_\lambda\|^2$ *for every* $\lambda \in \Gamma_i$, *where* $\|\ \|$ *is the norm of* $(\ ,\ )$.

2. *View the G-invariant almost complex structures on $M = G/K$ as endomorphisms J of square $-I$ on $\mathfrak{M}$ which commute with $ad_G(K)$. Such an endomorphism exists if and only if $\Gamma_i = -\Gamma_{-i}$ for all i,* i.e., $s = 0$. *If this condition is satisfied, then J is completely determined by the equations*

$$Jx_\lambda = \varepsilon(\lambda) y_\lambda\,, \qquad Jy_\lambda = -\varepsilon(\lambda) x_\lambda$$

for $\lambda \in \Delta - \Delta_K$. Here $\varepsilon(\lambda) = \pm 1$, $\varepsilon(-\lambda) = -\varepsilon(\lambda)$ for all $\lambda \in \Delta - \Delta_K$, and ε is constant on each Γ_i.

3. *Any G-invariant pseudo-riemannian metric* $(\ ,\ )$ *is compatible in the sense of* § 8 *with any G-invariant almost complex structure J, and hence they determine a G-invariant almost hermitian metric ds^2. In the notation of* (1) *and* (2) *above, the kaehler form F of ds^2 is the antisymmetric bilinear form on $\mathfrak{M}$ with the properties* (3a) $F(x_\lambda, x_\nu) = F(x_\lambda, y_\nu) = F(y_\lambda, y_\nu) = 0$ *for all* $\lambda, \nu \in \Delta - \Delta_K$ *with* $\lambda \neq \pm\nu$, *and* (3b) $F(x_\lambda, y_\lambda) = \varepsilon(\lambda)\|x_\lambda\|^2$ *for* $\lambda \in \Delta - \Delta_K$.

Proof. The $\mathfrak{N}_i$ are orthogonal because they are representation spaces for inequivalent representations of K. On $\mathfrak{N}_i$ the invariant bilinear form $(\ ,\ )$ must be proportional to the Killing form, which is negative definite; hence $(\ ,\ )$ is definite. Choose $\lambda \in \Gamma_i$ and define $c_i = \|x_\lambda\|^2$; now c_i is a nonzero real number and the first sentence of Lemma 9.2 shows that $c_i = \|y_\lambda\|^2$. If $\nu \in \Gamma_i$ with $\nu \neq \lambda$, then $\nu = \lambda + \sigma$ where $\sigma \in \Delta_K$. Using the $ad_G(K)$-invariance of $(\ ,\ )$ and Lemma 9.2, we compute

$$n_{\lambda,\sigma}\|x_\nu\|^2 = ([x_\lambda, x_\sigma], x_\nu) = (x_\lambda, [x_\sigma, x_\nu]) = n_{\nu,-\sigma}\|x_\lambda\|^2\,.$$

On the other hand, $0 \neq n_{\lambda,-\sigma} = n_{\nu,-\sigma}$ by (9.1) and so $\|x_\nu\|^2 = \|x_\lambda\|^2$. This proves (1).

Part (2) follows from Theorem 4.3. Compatibility is clear in (3), as is property (3a). We compute $F(x_\lambda, y_\lambda) = (Jx_\lambda, y_\lambda) = \varepsilon(\lambda)\|y_\lambda\|^2 = \varepsilon(\lambda)\|x_\lambda\|^2$, proving (3b). q.e.d.

Now we can characterize the classes $\mathscr{K}$, $\mathscr{A}\mathscr{K}$, and $\mathscr{H}$ for invariant almost hermitian metrics on $M = G/K$, rank K = rank G. Furthermore we characterize hodge metrics on M.

9.4. Theorem. *Let ds^2 be an invariant positive definite almost hermitian metric on $M = G/K$, where G is a compact Lie group and K is a subgroup of maximal rank. Let J and* $(\ ,\ )$ *be the almost complex structure and the riemannian metric associated to ds^2.*

1. *The following conditions are equivalent:*

(1a) $(M, ds^2) \in \mathscr{H}$ (i.e. *J is integrable*).

(1b) $\lambda, \nu, \lambda + \nu \in \Lambda - \Lambda_K$ *with* $\varepsilon(\lambda) = \varepsilon(\nu) = 1$ *implies* $\varepsilon(\lambda + \nu) = 1$.

(1c) *There exists a system of positive roots* Λ^+ *of* Λ *such that* $\Lambda^+ \cap \Lambda_K$ *is a system of positive roots for* Λ_K *and* $\Lambda^+ \cap (\Lambda - \Lambda_K) = \{\lambda \in \Lambda - \Lambda_K : \varepsilon(\lambda) = 1\}$.

2. *The following conditions are equivalent:*

(2a) *Let Z be the center of K. Then K is the centralizer of the torus* Z_0, *and there is a linear form* φ *on* $\mathfrak{T}^C$ *such that* $\langle\varphi, \lambda\rangle = 0$ for $\lambda \in \Lambda_K$ and $\langle\varphi, \lambda\rangle = \varepsilon(\lambda)\|x_\lambda\|^2$ for $\lambda \in \Lambda - \Lambda_K$.

(2b) *J is integrable, and* $\lambda, \nu, \lambda + \nu \in \Lambda - \Lambda_K$ *with* $\varepsilon(\lambda) = \varepsilon(\nu) = 1$ *implies* $\|x_{\lambda+\nu}\|^2 = \|x_\lambda\|^2 + \|x_\nu\|^2$.

(2c) $(M, ds^2) \in \mathscr{K}$.

(2d) $(M, ds^2) \in \mathscr{AK}$.

3. *Assume* $(M, ds^2) \in \mathscr{K}$. *Then the following conditions are equivalent:*

(3a) (M, ds^2) *is a hodge manifold.*

(3b) *If* φ *denotes the linear form defined in* (2a), *then the* $\langle\varphi, \lambda\rangle$ *are rational multiples of each other for* $\lambda \in \Lambda$.

(3c) *If* $\Psi = \{\psi_1, \cdots, \psi_l\}$ *is a simple system of roots of G such that* $\Psi_K = \{\psi_{r+1}, \cdots, \psi_l\}$ *is a simple system of roots of K, then* $\|x_{\psi_1}\|^2, \cdots, \|x_{\psi_r}\|^2$ *are rational multiples of each other.*

Proof. In the notation of Theorem 4.5, J has $(\sqrt{-1})$-eigenspace on $\mathfrak{M}^C$ given by $\mathfrak{M}^+ = \sum_{\substack{\lambda \in \Lambda - \Lambda_K \\ \varepsilon(\lambda) = 1}} \mathfrak{G}_\lambda$ and has $-(\sqrt{-1})$-eigenspace $\mathfrak{M}^- = \sum_{\substack{\lambda \in \Lambda - \Lambda_K \\ \varepsilon(\lambda) = -1}} \mathfrak{G}_\lambda$. Now $\mathfrak{K}^C + \mathfrak{M}^+$ is an algebra if and only if $\lambda, \nu, \lambda + \nu \in \Lambda - \Lambda_K$ with $\varepsilon(\lambda) = \varepsilon(\nu) = 1$ implies $\varepsilon(\lambda + \nu) = 1$, and Theorem 4.5 says that J is integrable if and only if $\mathfrak{K}^C + \mathfrak{M}^+$ is an algebra. This proves that (1a) and (1b) are equivalent.

It is clear that (1c) implies (1b), since the sum of positive roots is positive if it is a root. To prove the reverse implication we define Λ^+ to be the union of the positive roots Λ_K^+ of Λ_K and $\{\lambda \in \Lambda - \Lambda_K : \varepsilon(\lambda) = 1\}$. If $\lambda, \nu \in \Lambda^+$ and $\lambda + \nu$ is a root, then it follows from (1b) and $ad_G(K)$-invariance of J that $\lambda + \nu \in \Lambda^+$. Theorem 4.5 shows that Λ^+ is a system of positive roots of Λ.

Next we turn to (2) and prove that (2a) $\Rightarrow$ (2b) $\Rightarrow$ (2c) $\Rightarrow$ (2d) $\Rightarrow$ (2a). Let φ be the linear functional of (2a) and $\lambda, \nu, \lambda + \nu \in \Lambda - \Lambda_K$ be such that $\varepsilon(\lambda) = \varepsilon(\nu) = 1$. Then

$$\varepsilon(\lambda + \nu)\|x_{\lambda+\nu}\|^2 = \langle\varphi, \lambda + \nu\rangle = \langle\varphi, \lambda\rangle + \langle\varphi, \nu\rangle = \|x_\lambda\|^2 + \|x_\nu\|^2 .$$

Since (,) is positive definite, $\varepsilon(\lambda + \nu) = 1$ and $\|x_{\lambda+\nu}\|^2 = \|x_\lambda\|^2 + \|x_\nu\|^2$. From the equivalence of (1a) and (1b) it follows that J is interable. Hence (2a) implies (2b).

Next assume that (2b) holds. We define a linear form $\eta : \mathfrak{G} \to \boldsymbol{R}$ by $\eta(x_\lambda) = \eta(y_\lambda) = 0$ for $\lambda \in \Lambda$, $\eta(\sqrt{-1}\,h_\lambda) = 0$ for $\lambda \in \Lambda_K$, and $\eta(\sqrt{-1}\,h_\lambda) = -\frac{1}{2}\varepsilon(\lambda)\|x_\lambda\|^2$

for $\lambda \in \Delta - \Delta_K$. We view η as a left invariant 1-form on G. Then $d\eta(x_\lambda, y_\lambda) = -\eta([x_\lambda, y_\lambda]) = \varepsilon(\lambda)\|x_\lambda\|^2 = F(x_\lambda, y_\lambda)$ for $\lambda \in \Delta - \Delta_K$, and η vanishes on the rest of $\mathfrak{G} \times \mathfrak{G}$. Now the kaehler form F determines a 2-form π^*F on G, where $\pi: G \to M$ is the natural projection. The above calculation shows that $d\eta = \pi^*F$, and so $d\pi^*F = \pi^*dF = 0$. Since π^* is injective $dF = 0$. We are assuming that J is integrable and so $(M, ds^2) \in \mathscr{K}$. This proves that (2b) implies (2c).

Trivially (2c) implies (2d), and so it remains to show that (2d) implies (2a). If $(M, ds^2) \in \mathscr{A}\mathscr{K}$, then $dF = 0$, and so π^*F is closed, where $\pi: G \to M$ is the natural projection. The cohomology $H^2(\mathfrak{G}, \boldsymbol{R}) = 0$, and so it follows G has a left invariant real 1-form η with $d\eta = \pi^*F$. If $\lambda, \nu, \lambda + \nu \in \Delta - \Delta_K$, it follows, using (3b) of Proposition 9.4, that

$$\begin{aligned}
\varepsilon(\lambda + \nu)\|x_{\lambda+\nu}\|^2 &= (\pi^*F)(x_{\lambda+\nu}, y_{\lambda+\nu}) = -\eta([x_{\lambda+\nu}, y_{\lambda+\nu}]) \\
&= -\eta(2\sqrt{-1}\, h_{\lambda+\nu}) = -2\sqrt{-1}\langle \eta, \lambda + \nu\rangle \\
&= -2\sqrt{-1}\langle \eta, \lambda\rangle - 2\sqrt{-1}\langle \eta, \nu\rangle \\
&= -\eta([x_\lambda, y_\lambda]) - \eta([x_\nu, y_\nu]) \\
&= \varepsilon(\lambda)\|x_\lambda\|^2 + \varepsilon(\nu)\|x_\nu\|^2 .
\end{aligned}$$

As $(,)$ is positive definite, $\varepsilon(\lambda) = \varepsilon(\nu) = 1$ implies $\varepsilon(\lambda + \nu) = 1$. Hence by the equivalence of (1a) and (1b) it follows that J is integrable. By Theorem 4.5, K is the centralizer of Z_0. Furthermore, the above calculation shows that $\varphi = \frac{1}{2}\sqrt{-1}\,\eta$ satisfies the conditions of (2a). Thus (2d) implies (2a).

We prove (3). Let $(M, ds^2) \in \mathscr{K}$. By definition (M, ds^2) is a hodge manifold if and only if some nonzero real multiple of the de Rham cohomology class $[F] \in H^2(M, \boldsymbol{R})$ is an integral class. Let φ be the linear form defined in (2a); then φ is orthogonal to the roots of $\mathfrak{K}$ and (3b) is just the condition that some nonzero real multiple of φ exponentiate to a character ζ on K. So we must check that a nonzero multiple $a[F]$ is integral if and only if a nonzero multiple $b\varphi = \log \zeta$ for some character ζ on K. If exp $b\varphi$ is a character on K, then it induces a projective embedding of the complex manifold M as in [1, § 14.4], and a certain nonzero multiple $b[F]$ is the pull-back of the Chern class of the hyperplane section bundle; thus $b[F]$ is integral. If $b[F]$ is integral it is the Chern class of a positive line bundle $L \to M$; we may assume L homogeneous and find a G-invariant hermitian metric on it whose curvature form ω is a multiple of $[F]$, and then ω transgresses to a multiple $b\varphi \neq 0$; it follows that $\exp(b\varphi)$ is a well defined character on K. This proves equivalence of (3a) and (3b). Equivalence of (3b) and (3c) amounts to equivalence of (2a) and (2b). q.e.d.

We have the following consequence of Theorem 9.4, new for the class $\mathscr{A}\mathscr{K}$ and bringing together known results from various authors for the other classes.

9.5. Corollary. *Let $M = G/K$ where G is a compact Lie group and K is a subgroup of maximal rank.*

(i) *If K is the centralizer of a torus then there is a G-invariant almost hermitian metric ds^2 on M such that (M, ds^2) is a hodge manifold. In particular, $(M, ds^2) \in \mathscr{H}$, $(M, ds^2) \in \mathscr{A}\mathscr{K}$ and $(M, ds^2) \in \mathscr{K}$.*

(ii) *If K is not the centralizer of a torus, and ds^2 is any G-invariant almost hermitian metric on M, then $(M, ds^2) \notin \mathscr{H}$, $(M, ds^2) \notin \mathscr{A}\mathscr{K}$ and $(M, ds^2) \notin \mathscr{K}$.*

Proof. If K is not the centralizer of a torus then Theorem 4.5 shows $(M, ds^2) \notin \mathscr{H}$. In particular, $(M, ds^2) \notin \mathscr{K}$, and then Theorem 9.4 shows that $(M, ds^2) \notin \mathscr{A}\mathscr{K}$.

Let K be the centralizer of a torus. Theorem 4.5 gives us a system $\Psi = \{\psi_1, \cdots, \psi_l\}$ of simple roots of $\mathfrak{G}$ such that $\Psi_K = \{\psi_{r+1}, \cdots, \psi_l\}$ is a system of simple roots of $\mathfrak{K}$. We define J by: $\varepsilon(\lambda) = +1$ for $\lambda \in \Lambda^+ - \Lambda_K$, -1 for $-\lambda \in \Lambda^+ - \Lambda_K$, and the metric $(,)$ by $\|x_\lambda\|^2 = a_1 n_1 + \cdots + a_r n_r$ for $\lambda = \sum a_i \psi_i \in \Lambda^+ - \Lambda_K$, where the n_i are arbitrary positive integers. Then the ds^2 defined by J and $(,)$ is a hodge metric on M by Theorem 9.4. q.e.d.

For the classes $\mathscr{Q}\mathscr{K}$ and $\mathscr{N}\mathscr{K}$ we must look at the covariant derivatives of J. First, however, we need the following lemma. It is a long calculation, but it is straightforward from Lemma 9.2, and so we leave it to the reader.

9.6. Proposition. *Let $(,)$ be a pseudo-riemannian metric on a compact Lie group G which is invariant under left translation by G and right translation by the maximal torus T of G. Given $\lambda, \nu \in \Lambda$ with $\lambda \neq \pm\nu$, define numbers $a_{\lambda,\nu}$ and $b_{\lambda,\nu}$ by*

$$(9.7a) \qquad a_{\lambda,\nu} = \begin{cases} \dfrac{1}{2}\left(1 + \dfrac{\|x_\nu\|^2 - \|x_\lambda\|^2}{\|x_{\lambda+\nu}\|^2}\right) n_{\lambda,\nu} & \text{if } \lambda + \nu \in \Lambda, \\ 0 & \text{if } \lambda + \nu \notin \Lambda; \end{cases}$$

$$(9.7b) \qquad b_{\lambda,\nu} = \begin{cases} \dfrac{1}{2}\left(1 + \dfrac{\|x_\nu\|^2 - \|x_\lambda\|^2}{\|x_{\lambda-\nu}\|^2}\right) n_{\lambda,-\nu} & \text{if } \lambda - \nu \in \Lambda, \\ 0 & \text{if } \lambda - \nu \notin \Lambda. \end{cases}$$

View $\mathfrak{G}$ as the algebra of left invariant vector fields on G and let $\tilde{\nabla}$ be the riemannian connection of $(,)$. Then

$$(9.8a) \qquad \tilde{\nabla}_{x_\lambda}(x_\lambda) = \tilde{\nabla}_{y_\lambda}(y_\lambda) = 0 \quad \text{and} \quad \tilde{\nabla}_{x_\lambda}(y_\lambda), \tilde{\nabla}_{y_\lambda}(x_\lambda) \in \mathfrak{T};$$

$$(9.8b) \qquad \tilde{\nabla}_{x_\lambda}(x_\nu) = a_{\lambda,\nu} x_{\lambda+\nu} - b_{\lambda,\nu} x_{\lambda-\nu};$$

$$(9.8c) \qquad \tilde{\nabla}_{y_\lambda}(y_\nu) = -a_{\lambda,\nu} x_{\lambda+\nu} - b_{\lambda,\nu} x_{\lambda-\nu};$$

$$(9.8d) \qquad \tilde{\nabla}_{x_\lambda}(y_\nu) = a_{\lambda,\nu} y_{\lambda+\nu} + b_{\lambda,\nu} y_{\lambda-\nu};$$

$$(9.8e) \qquad \tilde{\nabla}_{y_\lambda}(x_\lambda) = a_{\lambda,\nu} y_{\lambda+\nu} - b_{\lambda,\nu} y_{\lambda-\nu}.$$

Now let ds^2 be a G-invariant almost hermitian metric on $M = G/K$. We lift its riemannian metric (,) to a riemannian metric on G which is left invariant by G, right invariant by K (thus also by the subgroup T of K), and also denoted by (,). (This metric is the sum of a bi-invariant metric on K and $\pi^*(,).$) We denote by ∇ and $\tilde{\nabla}$ the respective riemannian connections on M and G.

We lift the almost complex structure J of ds^2 to a tensor field $\tilde{J}$ on G as follows. If $g \in G$, then the tangent space G_g can be decomposed as $G_g = V_g \oplus H_g$ where $(V_g, H_g) = 0$ and $V_g = \text{kernel}\,(\pi_* | G_g)$. For $z \in G_g$ we may write uniquely $z = z_V + z_H$. If $w \in M_{\pi(g)}$, then $\tilde{w} \in G_g$ denotes the horizontal preimage under π_*. We set

$$\tilde{J}(z) = z_V + \widetilde{J(\pi_* z_H)} \quad \text{for } z \in G_g\,. \tag{9.9}$$

Then $\tilde{J}$ is a (1, 1) tensor field on G.

9.10. Lemma. *Let ds^2, (,), $\tilde{\nabla}$, and $\tilde{J}$ be as above, and use the notation (9.7). Then for $\lambda, \nu \in \Lambda - \Lambda_K$ with $\lambda \neq \pm\nu$ we have*

$$\begin{aligned}
&\tilde{\nabla}_{x_\lambda}(\tilde{J})(x_\lambda),\ \tilde{\nabla}_{x_\lambda}(\tilde{J})(y_\lambda),\ \tilde{\nabla}_{y_\lambda}(\tilde{J})(x_\lambda),\ \tilde{\nabla}_{y_\lambda}(\tilde{J})(y_\lambda) \in \mathfrak{T}\,,\\
&\tilde{\nabla}_{x_\lambda}(\tilde{J})(x_\nu) = a_{\lambda,\nu}(\varepsilon(\nu) - \varepsilon(\lambda+\nu))y_{\lambda+\nu} + b_{\lambda,\nu}(\varepsilon(\nu) + \varepsilon(\lambda-\nu))y_{\lambda-\nu}\,,\\
&\tilde{\nabla}_{y_\lambda}(\tilde{J})(y_\nu) = a_{\lambda,\nu}(-\varepsilon(\nu) + \varepsilon(\lambda+\nu))y_{\lambda+\nu} + b_{\lambda,\nu}(\varepsilon(\nu) + \varepsilon(\lambda-\nu))y_{\lambda-\nu}\,,\\
&\tilde{\nabla}_{x_\lambda}(\tilde{J})(y_\nu) = a_{\lambda,\nu}(-\varepsilon(\nu) + \varepsilon(\lambda+\nu))x_{\lambda+\nu} + b_{\lambda,\nu}(\varepsilon(\nu) + \varepsilon(\lambda-\nu))x_{\lambda-\nu}\,,\\
&\tilde{\nabla}_{y_\lambda}(\tilde{J})(x_\nu) = a_{\lambda,\nu}(-\varepsilon(\nu) + \varepsilon(\lambda+\nu))x_{\lambda+\nu} + b_{\lambda,\nu}(-\varepsilon(\nu) + \varepsilon(\lambda-\nu))x_{\lambda-\nu}\,.
\end{aligned}$$

Proof. These equations follow from Propositions 9.3 and 9.6, and the fact that (by definition) $\tilde{\nabla}_x(J)(y) = \tilde{\nabla}_x(\tilde{J}y) - J\tilde{\nabla}_x y$ for $x, y \in \mathfrak{G}$.

9.11. Proposition. *Let ds^2 be a G-invariant almost hermitian metric on G/K, and let $\lambda, \nu \in \Lambda - \Lambda_K$ with $\lambda \neq \pm\nu$. Then at the point of M at which K is the isotropy subgroup of G, we have*

$$\begin{aligned}
&\nabla_{x_\lambda}(J)(x_\lambda) = \nabla_{x_\lambda}(J)(y_\lambda) = \nabla_{y_\lambda}(J)(x_\lambda) = \nabla_{y_\lambda}(J)(y_\lambda) = 0\,,\\
&\nabla_{x_\lambda}(J)(x_\nu) = \{a_{\lambda,\nu}(\varepsilon(\nu) - \varepsilon(\lambda+\nu))y_{\lambda+\nu} + b_{\lambda,\nu}(\varepsilon(\nu) + \varepsilon(\lambda-\nu))y_{\lambda-\nu}\}_{\mathfrak{M}}\,,\\
&\nabla_{y_\lambda}(J)(y_\nu) = \{a_{\lambda,\nu}(-\varepsilon(\nu) + \varepsilon(\lambda+\nu))y_{\lambda+\nu} + b_{\lambda,\nu}(\varepsilon(\nu) + \varepsilon(\lambda-\nu))y_{\lambda-\nu}\}_{\mathfrak{M}}\,,\\
&\nabla_{x_\lambda}(J)(y_\nu) = \{a_{\lambda,\nu}(-\varepsilon(\nu) + \varepsilon(\lambda+\nu))x_{\lambda+\nu} + b_{\lambda,\nu}(\varepsilon(\nu) + \varepsilon(\lambda-\nu))x_{\lambda-\nu}\}_{\mathfrak{M}}\,,\\
&\nabla_{y_\lambda}(J)(x_\nu) = \{a_{\lambda,\nu}(-\varepsilon(\nu) + \varepsilon(\lambda+\nu))x_{\lambda+\nu} + b_{\lambda,\nu}(-\varepsilon(\nu) - \varepsilon(\lambda-\nu))x_{\lambda-\nu}\}_{\mathfrak{M}}\,.
\end{aligned}$$

Proof. Let $\pi: G \to M$ be the natural projection with $\pi(1) = m$, where 1 is the identity of G. There exists a coordinate neighborhood U of 1 in G mapping onto a coordinate neighborhood $\pi(U)$ of M such that U and $\pi(U)$ have the following property: each vector field X on $\pi(U)$ can be lifted to a horizontal vector field $\tilde{X}$ on U, i.e., $\pi_*(\tilde{X}) = X$ and $(\tilde{X}, \text{kernel}\,\pi_*) = 0$. There $\tilde{X}$ is not in general left invariant.

Now let $x, y \in M_m$ and choose vector fields X and Y on $\pi(U)$ such that $X_m = x$ and $Y_m = y$. If $\tilde{x}, \tilde{y} \in \mathfrak{G}$ are such that $\pi_*(\tilde{x}) = x$ and $\pi_*(\tilde{y}) = y$, then $\tilde{X}_1 = \tilde{x}$ and $\tilde{Y}_1 = \tilde{y}$. We have

$$(9.12) \qquad \tilde{J}\tilde{X} = \widetilde{JX}\,, \qquad \tilde{\nabla}_{\tilde{X}}\tilde{Y} = (\widetilde{\nabla_X Y})\,.$$

The first of these formulas easily follows from the definition of $\tilde{J}$, and the second is known [6, Theorem 3.2]. Note also that

$$(9.13) \qquad \nabla_x(J)(y) = \nabla_X(J)(Y)_m\,, \qquad \tilde{\nabla}_{\tilde{x}}(\tilde{J})(\tilde{y}) = \tilde{\nabla}_{\tilde{X}}(\tilde{J})(\tilde{Y})_1\,,$$

because $\nabla_X(J)(Y)$ and $\tilde{\nabla}_{\tilde{X}}(\tilde{J})(\tilde{Y})$ are tensorial in X and Y, and in $\tilde{X}$ and $\tilde{Y}$, respectively.

An easy computation from (9.12) and (9.13) shows that

$$(9.14) \qquad \pi_*\tilde{\nabla}_{\tilde{x}}(\tilde{J})(\tilde{y}) = \nabla_x(J)(Y)$$

for $x, y \in M_m$. If we apply π_* to each of the formulas in Lemma 9.10 and identify $x_\lambda, y_\lambda \in \mathfrak{M}$ with $\pi_*(x_\lambda), \pi_*(y_\lambda) \in M_m$, then using (9.14) we obtain all the formulas in the statement of Proposition 9.11. q.e.d.

In order to facilitate our consideration of the classes $\mathcal{QK}$ and $\mathcal{NK}$ we define tensor fields Q and N on M by the formulas

$$Q(x, y) = \nabla_x(J)(y) + \nabla_{Jx}(J)(Jy)\,,$$
$$N(x, y) = \nabla_x(J)(y) + \nabla_y(J)(x)\,,$$

where x and y are tangent vectors on M.

9.15. Theorem. *Let ds^2 be an invariant almost hermitian metric on $M = G/K$, where G is a compact Lie group and K is a subgroup of maximal rank. Then the following conditions are equivalent:*

(i) $(M, ds^2) \in \mathcal{QK}$.

(ii) *For all $\lambda, \nu \in \Lambda - \Lambda_K$ with $\lambda \neq \pm\nu$ we have $Q(x_\lambda, x_\nu) = Q(x_\lambda, y_\nu) = 0$.*

(iii) *For all $\lambda, \nu \in \Lambda - \Lambda_K$ such that $\lambda + \nu \in \Lambda - \Lambda_K$ and $\varepsilon(\lambda) = \varepsilon(\nu) = \varepsilon(\lambda + \nu)$, we have $\|x_{\lambda+\nu}\|^2 = \|x_\lambda\|^2 + \|x_\nu\|^2$.*

Proof. It is obvious that (i) implies (ii). Conversely. we always have $Q(x_\lambda, x_\lambda) = Q(x_\lambda, y_\lambda) = Q(y_\lambda, x_\lambda) = Q(y_\lambda, y_\lambda) = 0$, $Q(y_\lambda, y_\nu) = \varepsilon(\lambda)\varepsilon(\nu)Q(x_\lambda, x_\nu)$, and $Q(y_\lambda, x_\nu) = \varepsilon(\lambda)\varepsilon(\nu)Q(x_\lambda, y_\nu)$, for $\lambda, \nu \in \Lambda - \Lambda_K$, $\lambda \neq \pm\nu$. Hence (ii) implies (i).

To show the equivalence of (ii) and (iii) we first apply Proposition 9.11 to write $Q(x_\lambda, x_\nu)$ and $Q(x_\lambda, y_\nu)$:

$$(9.16a) \qquad \begin{aligned} Q(x_\lambda, x_\nu) = \{&a_{\lambda,\nu}[\varepsilon(\nu) - \varepsilon(\lambda) - \varepsilon(\lambda + \nu) + \varepsilon(\lambda)\varepsilon(\nu)\varepsilon(\lambda + \nu)]y_{\lambda+\nu} \\ &+ b_{\lambda,\nu}[\varepsilon(\nu) + \varepsilon(\lambda) + \varepsilon(\lambda - \nu) + \varepsilon(\lambda)\varepsilon(\nu)\varepsilon(\lambda - \nu)]y_{\lambda-\nu}\}_{\mathfrak{M}}\,, \end{aligned}$$

$$Q(x_\lambda, y_\nu) = \{a_{\lambda,\nu}[-\varepsilon(\nu) + \varepsilon(\lambda) + \varepsilon(\lambda+\nu) - \varepsilon(\lambda)\varepsilon(\nu)\varepsilon(\lambda+\nu)]x_{\lambda+\nu} + b_{\lambda,\nu}[\varepsilon(\nu) + \varepsilon(\lambda) + \varepsilon(\lambda-\nu) + \varepsilon(\lambda)\varepsilon(\nu)\varepsilon(\lambda-\nu)]x_{\lambda-\nu}\}_{\mathfrak{M}}. \tag{9.16b}$$

We first prove that (ii) implies (iii). Let $\lambda, \nu \in \Lambda - \Lambda_K$ be such that $\lambda + \nu \in \Lambda - \Lambda_K$ and $\varepsilon(\lambda) = \varepsilon(\nu) = \varepsilon(\lambda+\nu)$. Then by (ii), $Q(x_{\lambda+\nu}, x_\nu) = 0$. Since the two terms on the right hand side of (9.16a) are linearly independent and $(\lambda+\nu) - \nu = \lambda \in \Lambda - \Lambda_K$, by replacing λ by $\lambda+\nu$ in (9.16a) we have

$$0 = b_{\lambda+\nu,\nu}[\varepsilon(\nu) + \varepsilon(\lambda+\nu) + \varepsilon(\lambda) + \varepsilon(\lambda+\nu)\varepsilon(\nu)\varepsilon(\lambda)] = 4b_{\lambda+\nu,\nu}\varepsilon(\lambda).$$

Therefore $b_{\lambda+\nu,\nu} = 0$, and so $\|x_{\lambda+\nu}\|^2 = \|x_\lambda\|^2 + \|x_\nu\|^2$ by (9.7b).

Conversely, let $\lambda, \nu \in \Lambda - \Lambda_K$ be such that $\lambda \neq \pm\nu$. Without loss of generality we may assume that $\varepsilon(\lambda) = \varepsilon(\nu) = 1$. Then (9.16a) and (9.16b) reduce to

$$Q(x_\lambda, x_\nu) = 2\{b_{\lambda,\nu}[\varepsilon(\lambda) + \varepsilon(\lambda-\nu)]y_{\lambda-\nu}\}_{\mathfrak{M}},$$
$$Q(x_\lambda, y_\nu) = 2\{b_{\lambda,\nu}[\varepsilon(\lambda) + \varepsilon(\lambda-\nu)]x_{\lambda-\nu}\}_{\mathfrak{M}}.$$

Clearly $Q(x_\lambda, x_\nu)$ and $Q(x_\lambda, y_\nu)$ vanish if $\lambda - \nu \notin \Lambda - \Lambda_K$ or if $\lambda - \nu \in \Lambda - \Lambda_K$ and $\varepsilon(\lambda-\nu) = -1$. Suppose $\lambda - \nu \in \Lambda - \Lambda_K$ and $\varepsilon(\lambda-\nu) = +1$. Then by (iii) we have $\|x_{\lambda-\nu}\|^2 + \|x_\nu\|^2 = \|x_\lambda\|^2$. Hence $b_{\lambda,\nu} = 0$, and we again obtain $Q(x_\lambda, x_\nu) = Q(x_\lambda, y_\nu) = 0$. Thus (iii) implies (ii).

9.17. Theorem. *Let ds^2 be an invariant almost hermitian metric on $M = G/K$, where G is a compact Lie group and K is a subgroup of maximal rank. Then the following conditions are equivalent:*

(i) *$(M, ds^2) \in \mathcal{NK}$.*

(ii) *For all $\lambda, \nu \in \Lambda - \Lambda_K$ with $\lambda \neq \pm\nu$ we have $N(x_\lambda, x_\nu) = N(x_\lambda, y_\nu) = 0$.*

(iii) *For all $\lambda, \nu \in \Lambda - \Lambda_K$ such that $\lambda + \nu \in \Lambda - \Lambda_K$ and $\varepsilon(\lambda) = \varepsilon(\nu)$, we have $\|x_{\lambda+\nu}\|^2 = \|x_\lambda\|^2 + \|x_\nu\|^2$ if $\varepsilon(\lambda) = \varepsilon(\nu) = \varepsilon(\lambda+\nu)$, and $\|x_{\lambda+\nu}\|^2 = \|x_\lambda\|^2 = \|x_\nu\|^2$ if $\varepsilon(\lambda) = \varepsilon(\nu) = -\varepsilon(\lambda+\nu)$.*

Proof. It is obvious that (i) implies (ii). Conversely, we always have $N(x_\lambda, x_\lambda) = N(x_\lambda, y_\lambda) = N(y_\lambda, y_\lambda) = 0$ and $N(y_\lambda, y_\nu) = -\varepsilon(\lambda)\varepsilon(\nu)N(x_\lambda, x_\nu)$ for $\lambda, \nu \in \Lambda - \Lambda_K$ with $\lambda \neq \pm\nu$. Hence (ii) implies (i).

Proposition 9.11 gives us the following expression for $N(x_\lambda, x_\nu)$:

$$\left\{\frac{1}{2}n_{\lambda,\nu}\left[\varepsilon(\nu) - \varepsilon(\lambda) + (\varepsilon(\nu) + \varepsilon(\lambda) - 2\varepsilon(\lambda+\nu))\left(\frac{\|x_\nu\|^2 - \|x_\lambda\|^2}{\|x_{\lambda+\nu}\|^2}\right)\right]y_{\lambda+\nu} + \frac{1}{2}n_{\lambda,-\nu}\left[\varepsilon(\nu) + \varepsilon(\lambda) + (\varepsilon(\nu) - \varepsilon(\lambda) + 2\varepsilon(\lambda-\nu))\left(\frac{\|x_\nu\|^2 - \|x_\lambda\|^2}{\|x_{\lambda-\nu}\|^2}\right)\right]y_{\lambda-\nu}\right\}_{\mathfrak{M}}, \tag{9.18a}$$

Similarly, Proposition 9.11 gives us the following expression for $N(x_\lambda, y_\nu)$:

$$
\begin{aligned}
&\left\{\frac{1}{2} n_{\lambda,\nu}\Big[-\varepsilon(\nu) + \varepsilon(\lambda) + (-\varepsilon(\nu) - \varepsilon(\lambda)\right.\\
&\qquad + 2\varepsilon(\lambda+\nu))\left(\frac{\|x_\nu\|^2 - \|x_\lambda\|^2}{\|x_{\lambda+\nu}\|^2}\right)\Big] y_{\lambda+\nu}\\
&\qquad + \frac{1}{2} n_{\lambda,-\nu}\Big[\varepsilon(\nu) + \varepsilon(\lambda) + (\varepsilon(\nu) - \varepsilon(\lambda)\\
&\qquad\left. + 2\varepsilon(\lambda-\nu))\left(\frac{\|x_\nu\|^2 - \|x_\lambda\|^2}{\|x_{\lambda-\nu}\|^2}\right)\Big] y_{\lambda-\nu}\right\}_{\mathfrak{M}} ;
\end{aligned}
\tag{9.18b}
$$

We first prove that (ii) implies (iii). Let $\lambda, \nu \in \Lambda - \Lambda_K$ be such that $\lambda + \nu \in \Lambda - \Lambda_K$ and $\varepsilon(\lambda) = \varepsilon(\nu)$. Since $(M, ds^2) \in \mathscr{Q}\mathscr{K}$, we have that $\varepsilon(\lambda) = \varepsilon(\nu) = \varepsilon(\lambda+\nu)$ implies $\|x_{\lambda+\nu}\|^2 = \|x_\lambda\|^2 + \|x_\nu\|^2$. Suppose $\varepsilon(\lambda) = \varepsilon(\nu) = -\varepsilon(\lambda+\nu)$. Then by (ii), $N(x_\lambda, x_\nu) = 0$ and so from (9.18a) we obtain

$$
\begin{aligned}
0 &= \varepsilon(\nu) - \varepsilon(\lambda) + [\varepsilon(\nu) + \varepsilon(\lambda) - 2\varepsilon(\lambda+\nu)]\left(\frac{\|x_\lambda\|^2 - \|x_\nu\|^2}{\|x_{\lambda+\nu}\|^2}\right)\\
&= 4\varepsilon(\lambda)\left(\frac{\|x_\lambda\|^2 - \|x_\nu\|^2}{\|x_{\lambda+\nu}\|^2}\right) .
\end{aligned}
$$

Hence $\|x_\lambda\|^2 = \|x_\nu\|^2$. Furthermore $\varepsilon(\lambda+\nu) = \varepsilon(-\nu) = -\varepsilon(\lambda)$. Therefore the same argument with λ replaced by $\lambda+\nu$, ν replaced by $-\nu$, and $\lambda+\nu$ replaced by λ shows that $\|x_{\lambda+\nu}\|^2 = \|x_\nu\|^2$. Thus (ii) implies (iii).

Conversely, let $\lambda, \nu \in \Lambda - \Lambda_K$ be such that $\lambda \neq \pm\nu$. Without loss of generality we may assume that $\varepsilon(\lambda) = \varepsilon(\nu) = 1$. Then (9.18a) and (9.18b) reduce to

$$
\begin{aligned}
N(x_\lambda, x_\nu) = &\left\{n_{\lambda,\nu}[\varepsilon(\nu) - \varepsilon(\lambda+\nu)]\left(\frac{\|x_\nu\|^2 - \|x_\lambda\|^2}{\|x_{\lambda+\nu}\|^2}\right) y_{\lambda+\nu}\right.\\
&\left. + n_{\lambda,-\nu}\left[\varepsilon(\nu) + \varepsilon(\lambda-\nu)\left(\frac{\|x_\nu\|^2 - \|x_\lambda\|^2}{\|x_{\lambda-\nu}\|^2}\right)\right] y_{\lambda-\nu}\right\}_{\mathfrak{M}} ,
\end{aligned}
\tag{9.19a}
$$

$$
\begin{aligned}
N(x_\lambda, y_\nu) = &\left\{n_{\lambda,\nu}[-\varepsilon(\nu) + \varepsilon(\lambda+\nu)]\left(\frac{\|x_\nu\|^2 - \|x_\lambda\|^2}{\|x_{\lambda+\nu}\|^2}\right) y_{\lambda+\nu}\right.\\
&\left. + n_{\lambda,-\nu}\left[\varepsilon(\nu) + \varepsilon(\lambda-\nu)\left(\frac{\|x_\nu\|^2 - \|x_\lambda\|^2}{\|x_{\lambda-\nu}\|^2}\right)\right] y_{\lambda-\nu}\right\}_{\mathfrak{M}} .
\end{aligned}
\tag{9.19b}
$$

If $\lambda + \nu \notin \Lambda - \Lambda_K$, or if $\lambda + \nu \in \Lambda - \Lambda_K$ and $\varepsilon(\lambda+\nu) = 1$, it is easily checked that first terms on the right hand sides of (9.19a) and (9.19b) vanish. If $\lambda + \nu \in \Lambda - \Lambda_K$ and $\varepsilon(\lambda+\nu) = -1$, then by (iii) we have $\|x_\lambda\|^2 = \|x_\nu\|^2$. Hence the first terms on the right hand sides of (9.18a) and (9.18b) always vanish. Similarly the second terms vanish. Thus $N(x_\lambda, x_\nu) = N(x_\lambda, y_\nu) = 0$, and we have proved that (iii) implies (ii). q.e.d.

To illustrate the $\mathscr{N}\mathscr{K}$ criterion of Theorem 9.17, let $M=G/K$ where G is a compact connected centerless simple Lie group and K is a connected subgroup of maximal rank with center of order 3. Thus G/K is one of G_2/A_2, F_4/A_2A_2, $E_6/A_2A_2A_2$, E_7/A_2A_5, E_8/A_8 and E_8/A_2E_6. Let ds^2 be an invariant almost hermitian metric on M. Now $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$ as usual, and $\mathfrak{M}^C = \mathfrak{M}^+ + \mathfrak{M}^-$ (eigenspace decomposition under J) with $ad_G(K)$ irreducible on each of $\mathfrak{M}^\pm$. The brackets are

$$[\mathfrak{M}^+, \mathfrak{M}^+] = \mathfrak{M}^- , \quad [\mathfrak{M}^-, \mathfrak{M}^-] = \mathfrak{M}^+ , \quad [\mathfrak{M}^+, \mathfrak{M}^-] = \mathfrak{K}^C .$$

Let $\mathfrak{G}_\lambda, \mathfrak{G}_\nu \subset \mathfrak{M}^\pm$ such that $\lambda + \nu$ is a root. Now $\varepsilon(\lambda) = \varepsilon(\nu) = \pm 1 = -\varepsilon(\lambda + \nu)$. Thus the $\mathscr{Q}\mathscr{K}$ criterion is vacuous, and the $\mathscr{N}\mathscr{K}$ criterion is $\|x_\lambda\|^2 = \|x_\nu\|^2 = \|x_{\lambda+\nu}\|^2$, which again is automatic. Thus, just as asserted earlier in Theorem 8.15, we have $(M, ds^2) \in \mathscr{N}\mathscr{K} \subset \mathscr{Q}\mathscr{K}$.

To other spaces $M = G/K$, where G is a compact connected centerless simple Lie group, where K is a connected subgroup of maximal rank but is not the centralizer of a torus, and where M has $N > 0$ invariant almost complex structures, are (Theorem 4.11):

G/K	$E_7/A_2A_2A_2T^1$	E_8/A_4A_4	$E_8/A_2A_5T^1$
N	16	4	32
G/K	$E_8/A_2A_2A_2A_2$	$E_8/A_2A_2A_2A_1T^1$	$E_8A_2A_2A_2T^2$
N	16	256	8192

We apply the $\mathscr{N}\mathscr{K}$ and $\mathscr{Q}\mathscr{K}$ criteria to a few of them.

E_8/A_4A_4. Here K has center of order 5. Let z generate the center of K, and let $\mathfrak{M}_l$ denote the $e^{2\pi\sqrt{-1}l/5}$ eigenspace of $ad(z)$ on $\mathfrak{G}^C$. Then the decomposition of $\mathfrak{M}^C$ into irreducible representation spaces of $ad_G(K)$ is given by $\mathfrak{M}^C = \mathfrak{M}_1 + \mathfrak{M}_{-1} + \mathfrak{M}_2 + \mathfrak{M}_{-2}$, and we have $[\mathfrak{M}_i, \mathfrak{M}_j] \subset \mathfrak{M}_{i+j}$ taking subscripts modulo 5. If λ is a root with $\mathfrak{G}_\lambda \subset \mathfrak{M}_i$, then $\varepsilon(\mathfrak{M}_i)$ denotes $\varepsilon(\lambda)$ and $\|x_i\|$ denotes $\|x_\lambda\|$, relative to an invariant ds^2.

First consider the two invariant almost complex structures J with $\varepsilon(\mathfrak{M}_1) = \varepsilon(\mathfrak{M}_2)$. Complete such a J to an invariant ds^2, and compute:

bracket	$\mathscr{Q}\mathscr{K}$ condition	$\mathscr{N}\mathscr{K}$ condition
$[\mathfrak{M}_1, \mathfrak{M}_1]=\mathfrak{M}_2$	$\|x_2\|^2=2\|x_1\|^2$	$\|x_2\|^2=2\|x_1\|^2$
$[\mathfrak{M}_2, \mathfrak{M}_2]=\mathfrak{M}_{-1}$	none	$\|x_1\|^2=\|x_2\|^2$
$[\mathfrak{M}_1, \mathfrak{M}_2]=\mathfrak{M}_{-2}$	none	$\|x_1\|^2=\|x_2\|^2$

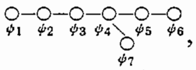
Thus $(M, ds^2) \notin \mathscr{N}\mathscr{K}$, and $(M, ds^2) \in \mathscr{Q}\mathscr{K}$ if and only if $\|x_2\|^2 = 2\|x_1\|^2$; in the latter case there is one real positive free parameter $\|x_1\|^2$ for ds^2.

Now consider the other two invariant almost complex structures J. They are given by $\varepsilon(\mathfrak{M}_1) = -\varepsilon(\mathfrak{M}_2)$. Complete such a J to an invariant ds^2 and compute:

bracket	$\mathcal{QK}$ condition	$\mathcal{NK}$ condition
$[\mathfrak{M}_1, \mathfrak{M}_1]=\mathfrak{M}_2$	none	$\|x_1\|^2=\|x_2\|^2$
$[\mathfrak{M}_{-2}, \mathfrak{M}_{-2}]=\mathfrak{M}_1$	$\|x_1\|^2=2\|x_2\|^2$	$\|x_1\|^2=2\|x_2\|^2$
$[\mathfrak{M}_1, \mathfrak{M}_{-2}]=\mathfrak{M}_{-1}$	none	$\|x_1\|^2=\|x_2\|^2$

Thus $(M, ds^2) \notin \mathcal{NK}$, and $(M, ds^2) \in \mathcal{QK}$ if and only if $\|x_1\|^2 = 2\|x_2\|^2$; in the latter case again $\|x_2\|^2$ is the free positive real parameter for ds^2. Now:

9.20. Proposition. *Let $M = E_8/A_4A_4$. Then $(M, ds^2) \notin \mathcal{NK}$ for every invariant almost hermitian metric on M. Let J be one of the four invariant almost complex structures on M. Then J is subordinate to an invariant almost hermitian metric ds^2 on M such that $(M, ds^2) \in \mathcal{QK}$, and any two such ds^2 are proportional.*

$E_7/A_2A_2A_2T^1$. Label the simple roots , so $\mathfrak{K}^C = \mathfrak{T}^C + \sum \mathfrak{G}_\lambda$ where the summation runs over all $\lambda = \sum a_i\psi_i$ such that $a_3 \equiv a_5 \equiv 0$ modulo 3. Now $\mathfrak{M}^C = \sum \mathfrak{M}_{ij}$, where $\mathfrak{M}_{ij}$ is the sum of all $\mathfrak{G}_\lambda$ with $\lambda = \sum a_i\psi_j$ and $(a_3, a_5) \equiv (i, j) \not\equiv (0, 0)$ modulo $(3, 3)$; the nonzero $\mathfrak{M}_{ij}$ are

$$\mathfrak{M}_{1,0} = \mathfrak{M}_{-2,-3}\,, \quad \mathfrak{M}_{0,1} = \mathfrak{M}_{-3,-2}\,, \quad \mathfrak{M}_{1,1} = \mathfrak{M}_{-2,-2}\,, \quad \mathfrak{M}_{2,1} = \mathfrak{M}_{-1,-2};$$
$$\mathfrak{M}_{-1,0} = \mathfrak{M}_{2,3}\,, \quad \mathfrak{M}_{0,-1} = \mathfrak{M}_{3,2}\,, \quad \mathfrak{M}_{-1,-1} = \mathfrak{M}_{2,2}; \quad \mathfrak{M}_{-2,-1} = \mathfrak{M}_{1,2}\,.$$

The bracket relations are: $[\mathfrak{M}_{i,j}, \mathfrak{M}_{r,s}] = \mathfrak{M}_{i+r,j+s}$ if the latter is nonzero, taking subscripts modulo $(3, 3)$.

Given an invariant ds^2, suppose that its almost complex structure is specified by

$$\varepsilon(\mathfrak{M}_{i_1,j_1}) = \varepsilon(\mathfrak{M}_{i_2,j_2}) = \varepsilon(\mathfrak{M}_{i_3,j_3}) = \varepsilon(\mathfrak{M}_{i_4,j_4}) = \pm 1\,.$$

Note $[\mathfrak{M}_{1,0}, \mathfrak{M}_{1,0}] = [\mathfrak{M}_{1,0}, \mathfrak{M}_{0,1}] = [\mathfrak{M}_{2,1}, \mathfrak{M}_{2,1}] = 0$ and their conjugates; $[\mathfrak{M}_{1,1}, \mathfrak{M}_{1,1}] = \mathfrak{M}_{-1,-1}$, and $[\mathfrak{M}_{-1,-1}, \mathfrak{M}_{-1,-1}] = \mathfrak{M}_{1,1}$. Thus the brackets $[\mathfrak{M}_{i_r,j_r}, \mathfrak{M}_{i_r,j_r}]$ result in neither an $\mathcal{NK}$ nor a $\mathcal{QK}$ condition. Now, looking for $\mathcal{NK}$ and $\mathcal{QK}$ conditions, we need only check that 6 brackets $[\mathfrak{M}_{i_r,j_r}, \mathfrak{M}_{i_s,j_s}]$, $1 \leq r < s \leq 4$.

Let $\varepsilon(\mathfrak{M}_{1,0}) = \varepsilon(\mathfrak{M}_{0,1}) = \varepsilon(\mathfrak{M}_{1,1}) = \varepsilon(\mathfrak{M}_{2,1})$. Then we see $(M, ds^2) \notin \mathcal{NK}$, and $(M, ds^2) \in \mathcal{QK}$ if and only if $\|x_{11}\|^2 = \|x_{10}\|^2 + \|x_{01}\|^2$ and $\|x_{21}\|^2 = 2\|x_{10}\|^2 + \|x_{01}\|^2$.

Let $\varepsilon(\mathfrak{M}_{1,0}) = \varepsilon(\mathfrak{M}_{0,1}) = \varepsilon(\mathfrak{M}_{1,1}) = \varepsilon(\mathfrak{M}_{-2,-1})$. Then we see $(M, ds^2) \notin \mathcal{NK}$, and $(M, ds^2) \in \mathcal{QK}$ if and only if $\|x_{11}\|^2 = \|x_{10}\|^2 + \|x_{01}\|^2$ and $\|x_{21}\|^2 = \|x_{10}\|^2 + 2\|x_{01}\|^2$.

Let $\varepsilon(\mathfrak{M}_{1,0}) = \varepsilon(\mathfrak{M}_{0,1}) = \varepsilon(\mathfrak{M}_{-1,-1}) = \varepsilon(\mathfrak{M}_{2,1})$. Then we see $(M, ds^2) \notin \mathcal{NK}$, and $(M, ds^2) \in \mathcal{QK}$ if and only if $\|x_{11}\|^2 = \|x_{01}\|^2 + \|x_{21}\|^2$ and $\|x_{10}\|^2 = \|x_{01}\|^2 + 2\|x_{21}\|^2$.

Let $\varepsilon(\mathfrak{M}_{1,0}) = \varepsilon(\mathfrak{M}_{0,1}) = \varepsilon(\mathfrak{M}_{-1,-1}) = \varepsilon(\mathfrak{M}_{-2,-1})$. Then we see $(M, ds^2) \notin \mathcal{NK}$, and $(M, ds^2) \in \mathcal{QK}$ if and only if $\|x_{11}\|^2 = \|x_{10}\|^2 + \|x_{21}\|^2$ and $\|x_{01}\|^2 = \|x_{10}\|^2 + 2\|x_{21}\|^2$.

Let $\varepsilon(\mathfrak{M}_{1,0}) = \varepsilon(\mathfrak{M}_{0,-1}) = \varepsilon(\mathfrak{M}_{1,1}) = \varepsilon(\mathfrak{M}_{2,1})$. Then we see $(M, ds^2) \notin \mathcal{NK}$ and $(M, ds^2) \notin \mathcal{QK}$.

Let $\varepsilon(\mathfrak{M}_{1,0}) = \varepsilon(\mathfrak{M}_{0,-1}) = \varepsilon(\mathfrak{M}_{1,1}) = \varepsilon(\mathfrak{M}_{-2,-1})$. Then we see $(M, ds^2) \notin \mathcal{NK}$, and $(M, ds^2) \in \mathcal{QK}$ if and only if $\|x_{11}\|^2 = \|x_{01}\|^2 + \|x_{21}\|^2$ and $\|x_{10}\|^2 = 2\|x_{01}\|^2$ $\|x_{21}\|^2$.

Let $\varepsilon(\mathfrak{M}_{1,0}) = \varepsilon(\mathfrak{M}_{0,-1}) = \varepsilon(\mathfrak{M}_{-1,-1}) = \varepsilon(\mathfrak{M}_{2,1})$. Then we see $(M, ds^2) \notin \mathcal{NK}$ and $(M, ds^2) \notin \mathcal{QK}$.

Let $\varepsilon(\mathfrak{M}_{1,0}) = \varepsilon(\mathfrak{M}_{0,-1}) = \varepsilon(\mathfrak{M}_{-1,-1}) = \varepsilon(\mathfrak{M}_{-2,-1})$. Then we see $(M, ds^2) \notin \mathcal{NK}$, and $(M, ds^2) \in \mathcal{QK}$ if and only if $\|x_{11}\|^2 = \|x_{10}\|^2 + \|x_{21}\|^2$ and $\|x_{01}\|^2 = 2\|x_{10}\|^2 + \|x_{21}\|^2$.

In summary, we have

9.21. Proposition. *Let $M = E_7/A_2A_2A_2T^1$. Then $(M, ds^2) \notin \mathcal{NK}$ for every invariant almost hermitian metric on M. Of the 16 invariant almost complex structures on M,*

(i) *4 have the property: if J is subordinate to an invariant almost hermitian metric ds^2 on M, then $(M, ds^2) \notin \mathcal{QK}$; and*

(ii) *12 have the property: the invariant almost hermitian metrics ds^2 to which J is subordinate, such that $(M, ds^2) \in \mathcal{QK}$, form a two-real-parameter family.*

$E_8/A_2A_2A_2A_2$. Here K has center $Z_3 \times Z_3$. Let $\{z_1, z_2\}$ generate that center. Then $\mathfrak{M}^C = \sum \mathfrak{M}_{s_1s_2}$ where $ad(z_i)$ is multiplication by $\exp(2\pi\sqrt{-1}\, s_i/3)$ on $\mathfrak{M}_{s_1s_2}$. The nonzero $\mathfrak{M}_{s_1s_2}$ are

$$\mathfrak{M}_{\pm1,0}, \quad \mathfrak{M}_{0,\pm1}, \quad \mathfrak{M}_{\pm1,\pm1} \quad \text{and} \quad \mathfrak{M}_{\pm1,\mp1},$$

and they are the irreducible representation spaces of $ad_G(K)$ on $\mathfrak{M}^C$. Obviously $[\mathfrak{M}_{s_1s_2}, \mathfrak{M}_{r_1r_2}] \subset \mathfrak{M}_{r_1+s_1, r_2+s_2}$, viewing $\mathfrak{K}^C$ as $\mathfrak{M}_{0,0}$ and taking subscripts modulo $(3, 3)$; if the bracket is nonzero and not in $\mathfrak{K}^C$, then the inclusion is equality by irreducibility of K.

Let L be the identity component of the centralizer of z_1 in G. Then $K \subset L \subset G$ forces L to be of type A_2E_6. $\mathfrak{L}^C = \mathfrak{K}^C + \mathfrak{M}_{0,1} + \mathfrak{M}_{0,-1}$ is generated by $\mathfrak{K}^C + \mathfrak{M}_{0,1}$ and acts irreducibly on $\mathfrak{M}_{1,0} + \mathfrak{M}_{1,1} + \mathfrak{M}_{1,-1}$ and on $\mathfrak{M}_{-1,0} + \mathfrak{M}_{-1,1} + \mathfrak{M}_{-1,-1}$. Thus $[\mathfrak{M}_{0,1}, \mathfrak{M}_{i,j}] = \mathfrak{M}_{i,j+1}$ for $(i, j) \not\equiv (0, -1)$. Similarly, using z_2, z_1z_2 and $z_1z_2^{-1}$, respectively, in place of z_1, we see that $\mathfrak{M}_{1,0}, \mathfrak{M}_{1,-1}$ and $\mathfrak{M}_{1,1}$ bracket surjectively. Now

$$(i, j) \not\equiv (-r, -s) \bmod (3, 3) \text{ implies } [\mathfrak{M}_{ij}, \mathfrak{M}_{rs}] = \mathfrak{M}_{i+r, j+s} .$$

Let ds^2 be an invariant almost hermitian metric. We may alter our original choice of z_1 and z_2 so that the almost complex structure of ds^2 is given by $\varepsilon(\mathfrak{M}_{1,0}) = \varepsilon(\mathfrak{M}_{0,1}) = \varepsilon(\mathfrak{M}_{1,1}) = \varepsilon(\mathfrak{M}_{1,-1}) = 1$. The $\mathcal{QK}$ condition for three of the brackets is

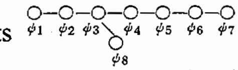
(i) $[\mathfrak{M}_{1,0}, \mathfrak{M}_{0,1}] = \mathfrak{M}_{1,1} : \|x_{1,1}\|^2 = \|x_{1,0}\|^2 + \|x_{0,1}\|^2$,

(ii) $[\mathfrak{M}_{0,1}, \mathfrak{M}_{1,1}] = \mathfrak{M}_{1,-1} : \|x_{1,-1}\|^2 = \|x_{0,1}\|^2 + \|x_{1,1}\|^2$,

(iii) $[\mathfrak{M}_{0,1}, \mathfrak{M}_{1,-1}] = \mathfrak{M}_{1,0} : \|x_{1,0}\|^2 = \|x_{0,1}\|^2 + \|x_{1,-1}\|^2$.

That gives $\|x_{1,0}\|^2 > \|x_{1,-1}\|^2$ (by (iii)) $> \|x_{1,1}\|^2$ (by (ii)) $> \|x_{1,0}\|^2$ (by (i)), which is absurd. Thus $(M, ds^2) \notin \mathcal{QK}$. In summary, we have

9.22. Proposition. *Let $M = E_8/A_2A_2A_2A_2$, and ds^2 be an invariant almost hermitian metric on M. Then $(M, ds^2) \notin \mathcal{QK}$. In particular $(M, ds^2) \notin \mathcal{NK}$.*

$E_8/A_1A_5T^1$. Label the simple roots  ; so $\mathfrak{K}^C = \mathfrak{T}^C + \sum \mathfrak{G}_\lambda$ where the the summation runs over all $\lambda = \sum a_i\psi_i$ with $(a_8, a_3) \equiv (0, 0)$ modulo $(3, 6)$. Then $\mathfrak{M}^C = \sum \mathfrak{M}_{ij}$ is the decomposition into irreducible representation spaces, where $\mathfrak{M}_{ij}$ is the sum of all $\mathfrak{G}_\lambda$, $\lambda = \sum a_r\psi_r$, such that $(a_8, a_5) \equiv (i, j) \not\equiv (0, 0)$ modulo $(3, 6)$. The $\mathfrak{M}_{ij}$ are

$$\mathfrak{M}_{1,0}, \quad \mathfrak{M}_{0,1} = \mathfrak{M}_{-3,-5}, \quad \mathfrak{M}_{1,1} = \mathfrak{M}_{-2,-5}, \quad \mathfrak{M}_{1,2} = \mathfrak{M}_{-2,-4}, \quad \mathfrak{M}_{1,3} = \mathfrak{M}_{-2,-3},$$
$$\mathfrak{M}_{-1,0}, \quad \mathfrak{M}_{0,-1} = \mathfrak{M}_{3,5}, \quad \mathfrak{M}_{-1,-1} = \mathfrak{M}_{2,5}, \quad \mathfrak{M}_{-1,-2} = \mathfrak{M}_{2,4}, \quad \mathfrak{M}_{-1,-3} = \mathfrak{M}_{2,3};$$

this is seen from a list of roots of $\mathfrak{E}_8$. A calculation, which is straightforward but too long to reproduce here, now shows

9.23. Proposition. *Let $M = E_8/A_2A_5T^1$, and ds^2 be an invariant almost hermitian metric on M. Then $(M, ds^2) \notin \mathcal{NK}$.*

We leave it to the reader to decide whether any of the 256 invariant almost complex structures on $E_8/A_2A_2A_2A_1T^1$, or any of the 8192 invariant almost complex structures on $E_8/A_2A_2A_2T^2$, is subordinate to an invariant ds^2 which is nearly kaehlerian.

The existence question for quasi-kaehlerian metrics is easier; there we will prove

9.24. Theorem. *Let $M = G/K$, where G is a compact connected Lie group acting effectively, K is a subgroup of maximal rank, and M admits G-invariant almost complex structures. Decompose*

$$G = G_1 \times \cdots \times G_r , \quad K = K_1 \times \cdots \times K_r , \quad M = M_1 \times \cdots \times M_r ,$$

where the G_i are the simple normal subgroups of G, $K_i = K \cap G_i$ and $M_i = G_i/K_i$.

1. *The following conditions are equivalent:*

(1a) *There is a G-invariant almost hermitian metric ds^2 on M such that $(M, ds^2) \in \mathcal{QK}$.*

(1b) *$\mathfrak{K} = \mathfrak{G}^\theta$ for some automorphism θ of odd order on $\mathfrak{G}$.*

(1c) *$G_i/K_i \neq E_8/A_2A_2A_2A_2$ for some index i, $1 \leq i \leq r$.*

2. *Assume the conditions of* (1). *Then M carries a G-invariant almost hermitian metric ds^2 such that $(M, ds^2) \in \mathcal{QK}$ and $(M, ds^2) \notin \mathcal{K}$, if and only if $M = G/K$ is not a hermitian symmetric coset space.*

3. *Assume the conditions of* (1). *Then M carries a G-invariant almost hermitian metric ds^2 such that $(M, ds^2) \in \mathcal{QK}$ and $(M, ds^2) \notin \mathcal{NK}$, if and only if there is an index i, $1 \leq i \leq r$, such that*

(i) *$M_i = G_i/K_i$ is not a hermitian symmetric space, and*

(ii) *the center of K_i does not have order* 3, i.e. *G_i/K_i is not one of G_2/A_2, F_4/A_2A_2, $E_6/A_2A_2A_2$, E_7/A_2A_5, E_8/A_8, E_8/A_2E_6.*

Proof. The theorem is valid for G/K if and only if it is valid for each of the G_i/K_i. Now we may assume G simple.

If K is the centralizer of a toral subgroup of G, then both (1b) and (1c) are immediate. If K is not the centralizer of a torus, then equivalence of (1b) and (1c) is contained in the statement of Theorem 4.10. Proposition 9.22 shows that (1a) implies (1c). The proof of statement 1 is now reduced to the proof that (1b) implies (1a).

Let $\mathfrak{K} = \mathfrak{G}^\theta$ where θ has odd order $k = 2u + 1$, $u \geq 1$. Then at least one of the eigenvalues of θ is a primitive k-th root η of 1. Let $\mathfrak{M}_n$ denote η^n-eigenspace of θ on $\mathfrak{G}^C$. Then

$$\mathfrak{G} = \mathfrak{K} + \mathfrak{M} \text{ where } \mathfrak{M}^C = \sum_{s=1}^{u}(\mathfrak{M}_s + \mathfrak{M}_{-s}) . \tag{9.25a}$$

It may happen that some of the $\mathfrak{M}_{\pm s}$ are 0, but at least $\mathfrak{M}_{\pm 1} \neq 0$. Now we define an invariant almost complex structure J on M by

$$\varepsilon(\lambda) = 1 \text{ if and only if } \mathfrak{G}_\lambda \subset \mathfrak{M}_1 + \cdots + \mathfrak{M}_u . \tag{9.25b}$$

In other words, $\mathfrak{M}_1 + \cdots + \mathfrak{M}_u$ is the $(\sqrt{-1})$-eigenspace of J and $\mathfrak{M}_{-1} + \cdots + \mathfrak{M}_{-u}$ is the $(-\sqrt{-1})$-eigenspace. Finally we define a G-invariant riemannian metric (,) on M by

$$\|x_\lambda\|^2 = \|y_\lambda\|^2 = s \text{ for } \mathfrak{G}_\lambda \subset \mathfrak{M}_s + \mathfrak{M}_{-s} ,\ 1 \leq s \leq u . \tag{9.25c}$$

ds^2 denotes the G-invariant almost hermitian metric on M defined by the data (9.25).

Suppose that we have roots $\lambda, \nu, \lambda + \nu \in \Lambda - \Lambda_K$ with $\varepsilon(\lambda) = \varepsilon(\nu) = \varepsilon(\lambda + \nu)$. If these signs are $+1$ then $\mathfrak{G}_\lambda \subset \mathfrak{M}_s$, $\mathfrak{G}_\nu \subset \mathfrak{M}_t$ and $\mathfrak{G}_{\lambda+\nu} \subset \mathfrak{M}_{s+t}$ where $1 \leq s \leq u$, $1 \leq t \leq u$ and $1 \leq s + t \leq u$. Now

$$\|x_{\lambda+\nu}\|^2 = s + t = \|x_\lambda\|^2 + \|x_\nu\|^2 .$$

If the signs are -1 we replace $\lambda, \nu, \lambda + \nu$ by their negatives and get the same result. Now $(M, ds^2) \in \mathcal{QK}$ by Theorem 9.15. This completes the proof of statement 1.

$(M, ds^2) \in \mathcal{NK}$ if and only if $\varepsilon(\lambda) = \varepsilon(\nu) = -\varepsilon(\lambda + \nu)$ implies $\|x_\lambda\|^2 = \|x_\nu\|^2 = \|x_{\lambda+\nu}\|^2$, for we already have $(M, ds^2) \in \mathcal{QK}$. It suffices to check the case $\varepsilon(\lambda) = 1$ and $\|x_\lambda\|^2 \leq \|x_\nu\|^2$, i.e. the case where $\mathfrak{G}_\lambda \subset \mathfrak{M}_s$ and $\mathfrak{G}_\nu \subset \mathfrak{M}_t$ with $1 \leq s \leq t \leq u$. If $s + t > u$, so $\mathfrak{G}_{\lambda+\nu} \subset \mathfrak{M}_{k-s-t}$, (9.25c) shows that the $\mathcal{NK}$ condition is $s = t = k - s - t$. In summary, we have

9.26. Proposition. *Let ds^2 be defined by* (9.25). *Then $(M, ds^2) \in \mathcal{QK}$, and $(M, ds^2) \in \mathcal{NK}$ if and only if $1 \leq s \leq t \leq u$ and $[\mathfrak{M}_s, \mathfrak{M}_t] \neq 0$ implies that either $s + t \leq u$ or $3s = k = 3t$. In particular, if k is not divisible by* 3 *then $(M, ds^2) \in \mathcal{NK}$ if and only if $(M, ds^2) \in \mathcal{K}$.*

We prove statements 2 and 3 for the case where K is not the centralizer of a torus. By Propositions 9.20, 9.21, 9.22 and 9.23, it suffices to consider the cases (i) $G/K = E_8/A_2A_2A_2A_1T^1$ and (ii) $G/K = E_8/A_2A_2A_2T^2$; in those cases we must prove that there exists an invariant quasi-kaehlerian ds^2 which is not nearly kaehlerian. So we assume that every quasi-kaehlerian ds^2 is nearly kaehlerian and find a contradiction.

To do this, note that Theorem 4.10 allows us to assume $k = 9$ in case (i) (set $n_3 = n_6 = 1$) and $k = 27$ in case (ii) (set $n_3 = 1$, $n_6 = 2$, $n_8 = 5$). Then $k = 3l$ with l divisible by 3. Define $\varphi = \theta^l$ and $\mathfrak{L} = \mathfrak{G}^\varphi$. Then φ has order 3 and $G = E_8$; so the analytic subgroup L is A_2E_6 or A_8, and G/L has no invariant complex structures. Let $\mathfrak{N}$ be the complement to $\mathfrak{L}$ in $\mathfrak{G}$;

$$\mathfrak{N}^C = \mathfrak{N}^+ + \mathfrak{N}^-, \quad \mathfrak{N}^+ = \sum_{\substack{s=1 \\ s \not\equiv 0(3)}}^{u} \mathfrak{M}_s, \quad \mathfrak{N}^- = \sum_{\substack{s=1 \\ s \not\equiv 0(3)}}^{u} \mathfrak{M}_{-s}\,.$$

Then Proposition 9.26 says that $\mathfrak{N}^\pm$ are algebras, for $\mathfrak{M}_l + \mathfrak{M}_{-l} \subset \mathfrak{L}^C$. As G/L has no invariant complex structure it follows that $ad_G(L)$ cannot normalize $\mathfrak{N}^+$, i.e. that $[\mathfrak{L}^C, \mathfrak{N}^+] \not\subset \mathfrak{N}^+$. As $\mathfrak{L}^C = \mathfrak{K}^C + \sum_{s \equiv 0(3)} \mathfrak{M}_s + \mathfrak{M}_{-s}$, this says that there exist indices s and t, $1 \leq t \leq s \leq u$, s divisible by 3 and t prime to 3, such that $[\mathfrak{M}_{-s}, \mathfrak{M}_t] \neq 0$. Our contradiction will consist of showing $[\mathfrak{M}_s, \mathfrak{M}_t] = 0$ for s divisible by 3 and t prime to 3. Replacing θ by a power θ^v, v prime to k, it suffices to show $[\mathfrak{M}_s, \mathfrak{M}_u] = 0$ for s divisible by 3.

In case (i), $k = 9$, $l = 3$ and $u = 4$. We apply Proposition 9.26 to the ds^2 defined by θ to see $[\mathfrak{M}_3, \mathfrak{M}_4] = 0$, to the ds^2 defined by θ^4 to see $[\mathfrak{M}_{-3}, \mathfrak{M}_4] = 0$. That gives us our contradiction.

In case (ii), $k = 27$, $l = 9$ and $u = 13$. We apply Proposition 9.26 to the ds^2 defined by θ to see $[\mathfrak{M}_3, \mathfrak{M}_{13}] = [\mathfrak{M}_6, \mathfrak{M}_{13}] = [\mathfrak{M}_9, \mathfrak{M}_{13}] = [\mathfrak{M}_{12}, \mathfrak{M}_{13}] = 0$, to the ds^2 defined by θ^4 to see $[\mathfrak{M}_{-3}, \mathfrak{M}_{13}] = 0$, to the ds^2 defined by θ^7 to see $[\mathfrak{M}_{-6}, \mathfrak{M}_{13}] = 0$, to the ds^2 defined by θ^8 to see $[\mathfrak{M}_{-12}, \mathfrak{M}_{13}] = 0$, and to the ds^2 defined by θ^{10} to see $[\mathfrak{M}_{-9}, \mathfrak{M}_{13}] = 0$. That gives the contradiction.

Finally we prove statements 2 and 3 for the case where K is the centralizer of a torus. We take a simple root system $\Psi = \Psi_K \cup \{\psi_1, \cdots, \psi_v\}$ of G where Ψ_K is a simple root system of K. $\mu = m_1\psi_1 + \cdots + m_v\psi_v + \kappa$, κ a linear combination of elements of Ψ_K, is the maximal root. If $\lambda \in \Lambda - \Lambda_K$ then $\mathfrak{M}_\lambda$ denotes the $ad_G(K)$-irreducible subspace of $\mathfrak{M}^C$ which contains $\mathfrak{G}_\lambda$. We define an invariant riemannian metric on M by

$$\|x_\lambda\|^2 = |a_1| + \cdots + |a_v| \quad \text{for} \quad \lambda = a_1\psi_1 + \cdots + a_v\psi_v + \kappa \in \Lambda - \Lambda_K .$$

We define in invariant almost complex structure on M by

$$\varepsilon(\mu) = -1, \text{ and } \varepsilon(\lambda) = +1 \text{ for } \lambda \in \Lambda^+ - \Lambda_K \text{ with } \mathfrak{M}_\lambda \neq \mathfrak{M}_\mu .$$

Let ds^2 denote the resulting invariant almost hermitian metric.

Let $\lambda, \nu, \lambda + \nu \in \Lambda - \Lambda_K$ with $\varepsilon(\lambda) = \varepsilon(\nu) = 1$. First suppose $\mathfrak{M}_\lambda \neq \mathfrak{M}_{-\mu} \neq \mathfrak{M}_\nu$, i.e. $\lambda, \nu \in \Lambda^+$; then $\lambda + \nu \in \Lambda^+$. If $\varepsilon(\lambda + \nu) = 1$ then the $\mathscr{Q}\mathscr{K}$ and $\mathscr{N}\mathscr{K}$ conditions are $\|x_{\lambda+\nu}\|^2 = \|x_\lambda\|^2 + \|x_\nu\|^2$, which is automatic. If $\varepsilon(\lambda + \nu) = -1$, i.e. if $\mathfrak{M}_{\lambda+\nu} = \mathfrak{M}_\mu$, then there is no $\mathscr{Q}\mathscr{K}$ condition, and the $\mathscr{N}\mathscr{K}$ condition is $\|x_\lambda\|^2 = \|x_\mu\|^2 = \|x_\nu\|^2$, which is impossible. Next suppose $\mathfrak{M}_\lambda \neq \mathfrak{M}_{-\mu} = \mathfrak{M}_\nu$. Then $\lambda + \nu \in \Lambda^-$ and $\mathfrak{M}_{\lambda+\nu} \neq \mathfrak{M}_{-\mu}$, so $\varepsilon(\lambda + \nu) = -1$ and there is no $\mathscr{Q}\mathscr{K}$ condition. Finally note that we cannot have $\mathfrak{M}_\lambda = \mathfrak{M}_{-\mu} = \mathfrak{M}_\nu$ because $[\mathfrak{M}_{-\mu}, \mathfrak{M}_{-\mu}] = 0$. Thus $(M, ds^2) \in \mathscr{Q}\mathscr{K}$, and $(M, ds^2) \in \mathscr{N}\mathscr{K}$ if and only if $\lambda, \nu \in \Lambda^+ - \Lambda_K$ with $\lambda + \nu = \mu$ is impossible, i.e. if and only if $v = 1 = m_1$, i.e. if and only if $M = G/K$ is hermitian symmetric. q.e.d.

Finally we come to the problem of deciding which $M = G/K$ admit invariant almost hermitian metrics ds^2 such that $(M, ds^2) \in \mathscr{N}\mathscr{K}$ but $(M, ds^2) \notin \mathscr{K}$. Here we are assuming G compact, connected and effective, and rank $G =$ rank K, so the problem comes down to the case where G is simple. If K is not the centralizer of a torus then $(M, ds^2) \notin \mathscr{K}$ is automatic; the problem is open for $E_8/A_2A_2A_2A_1T^1$ and $E_8/A_2A_2A_2T^2$, and in the other cases we know that the following conditions are equivalent:

(i) M admits a G-invariant almost hermitian metric ds^2 such that $(M, ds^2) \in \mathscr{N}\mathscr{K}$.

(ii) Every G-invariant almost hermitian metric ds^2 on M satisfies $(M, ds^2) \in \mathscr{N}\mathscr{K}$.

(iii) The center of K has order 3.

(iv) $\mathfrak{K} = \mathfrak{G}^\theta$ for an automorphism θ of order 3.

(v) G/K is G_2/A_2, E_4/A_2A_2, $E_6/A_2A_2A_2$, E_7/A_2A_5, E_8/A_8 or E_8/A_2E_6.

Now suppose that K is the centralizer of a torus. Choose a simple root system $\Psi = \Psi_K \cup \{\psi_1, \cdots, \psi_r\}$ of G where Ψ_K is a simple root system for K. We are looking for an invariant almost hermitian metric ds^2 on $M = G/K$ such that $(M, ds^2) \in \mathscr{N}\mathscr{K}$ but $(M, ds^2) \notin \mathscr{K}$. As before, $\mu = m_1\psi_1 + \cdots + m_r\psi_r + \kappa$, κ a linear combination of elements of Ψ_K, is the maximal root. Note $r = \dim H^2(M; \boldsymbol{R})$ from the homotopy sequence $\pi_2(G) \to \pi_2(M) \to \pi_1(K) \to \pi(G)$ and the Hurewicz isomorphism $\pi_2(M) \cong H_2(M; \boldsymbol{Z})$.

Let $r = 1$. Given an integer s, $0 \leq |s| \leq m_1$, $\mathfrak{M}_s$ denotes the $ad_G(K)$-irreducible subspace of $\mathfrak{M}^C$ which is the sum of all $\mathfrak{G}_\lambda$ with λ of the form $s\psi_1 + \kappa$, $\|x_s\|^2$ denotes $\|x_\lambda\|^2$, and $\varepsilon(s\psi_1)$ denotes $\varepsilon(\lambda)$ for λ of that form.

$m_1 = 1$ is the hermitian symmetric case.

$m_1 = 2$. There $\|x_1\|^2 = \|x_2\|^2$, $\varepsilon(\psi_1) = -\varepsilon(2\psi_1)$, defines two 1-parameter families of invariant ds^2 such that $(M, ds^2) \in \mathcal{N}\mathcal{K}$ but $(M, ds^2) \notin \mathcal{K}$.

$m_1 \geq 3$. Suppose $(M, ds^2) \in \mathcal{N}\mathcal{K}$ and let $1 < s \leq m_1$. Our induction hypothesis is $\varepsilon(\psi_1) = \varepsilon(t\psi_1)$ for $1 \leq t < s$; so the $\mathcal{N}\mathcal{K}$ condition implies $\|x_t\|^2 = t\|x_1\|^2$ for $1 \leq t < s$. Suppose $\varepsilon(s\psi_1) = -\varepsilon(\psi_1)$. If $1 \leq t_1 \leq t_2$ and $t_1 + t_2 = s$, then the $\mathcal{N}\mathcal{K}$ condition says $\|x_{t_1}\|^2 = \|x_s\|^2 = \|x_{t_2}\|^2$. In particular, we may take $t_1 = 1$ and $t_2 = s - 1$ and conclude $s = 2$. Now we are reduced to considering the case $\varepsilon(\psi_1) = -\varepsilon(2\psi_1)$ where the $\mathcal{N}\mathcal{K}$ condition says $\|x_1\|^2 = \|x_2\|^2$. If $\varepsilon(3\psi_1) = \varepsilon(\psi_1)$ we get $\|x_1\|^2 = \|x_2\|^2 + \|x_3\|^2$; if $\varepsilon(3\psi_1) = -\varepsilon(\psi_1)$ we get $\|x_2\|^2 = \|x_1\|^2 + \|x_3\|^2$; both are inconsistent with $\|x_1\|^2 = \|x_2\|^2$. Thus $(M, ds^2) \in \mathcal{N}\mathcal{K}$ implies $\varepsilon(\psi_1) = \varepsilon(2\psi_1) = \cdots = \varepsilon(m_1\psi_1)$ and $\|x_t\|^2 = t\|x_1\|^2$ for $1 \leq t \leq m_1$, which in turn says $(M, ds^2) \in \mathcal{K}$.

Phrasing in terms of automorphisms we summarize as follows:

9.27. Proposition. *Suppose $r = 1$. Then M has an invariant ds^2 such that $(M, ds^2) \in \mathcal{N}\mathcal{K}$ but $(M, ds^2) \notin \mathcal{K}$, if and only if* (i) *$M = G/K$ is not a hermitian symmetric coset space and* (ii) *$\mathfrak{K} = \mathfrak{G}^\theta$ for some automorphism θ of order* 3.

The case $r = 2$ is considerably more difficult, and we have not been able to settle it except in the case where K is a maximal torus of G. There one has only the possibilities (i) $\boldsymbol{A}_2/\boldsymbol{T}^2$, (ii) $\boldsymbol{B}_2/\boldsymbol{T}^2$ and (iii) $\boldsymbol{G}_2/\boldsymbol{T}^2$ for G/K, and (i) is the only one for which $\mathfrak{K} = \mathfrak{G}^\theta$ where θ is an automorphism of order 3. All this "evidence" adds up to

9.28. Conjecture. *Let $M = G/K$, where G is a compact connected Lie group acting effectively, K is a subgroup of maximal rank, and M carries a G-invariant almost complex structure. Suppose that $M = G/K$ is not a hermitian symmetric coset space. Then there is an invariant almost hermitian metric ds^2 such that $(M, ds^2) \in \mathcal{N}\mathcal{K}$ and $(M, ds^2) \notin \mathcal{K}$ if and only if $\mathfrak{K} = \mathfrak{G}^\theta$ for some automorphism θ of order 3 on $\mathfrak{G}$.*

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